AMATH 231: Assignment 4

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3. $\int\int \left|\left|\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}\right|\right| \, du\, dv = \int_0^{2\pi}\int_0^1 ||(\cos v, \sin v, 0) \times (-u \sin v, u\cos v, 1)|| \, du\, dv =$
$\int_0^{2\pi}\int_0^1 ||(\sin v, -\cos v, u)|| \, du\, dv = \int_0^{2\pi}\int_0^1 \sqrt{1+u^2}\, du\, dv = \left(\int_0^{2\pi} dv\right)\left(\int_0^1 \sqrt{1+u^2}\, du\right) =$
$2\pi\left(\int_0^1 \sqrt{1+u^2}\, du\right) = 2\pi\frac{1}{2}\left(x\sqrt{x^2+1} + \ln|x + \sqrt{x^2+1}|\right)_0^1 = \pi(\sqrt{2} + \ln(1+\sqrt{2}))$

6. **(a)** The parametric vector of the circle $(y-b)^2 + z^2 = a^2$ with $x = 0$ is $\mathbf{f} = [0, b + a\cos u, a \sin u]^T$ for $0 \le u \le 2\pi$. The rotation matrix R_z for counterclockwise rotation about the z-axis with angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the parametric form of the torus is simply $R_z\mathbf{f} = (-(b + a\cos u)\sin\theta, (b + a\cos u)\cos\theta, a\sin u)$ over $0 \le u \le 2\pi$ and $0 \le \theta \le 2\pi$ which is equivalent to $\mathbf{g}(u, v)$ over $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ with the substitution $v - \frac{\pi}{2} = \theta$.

(b) $\int\int \left|\left|\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}\right|\right| \, du\, dv =$
$\int_0^{2\pi}\int_0^{2\pi} ||(-a\sin u\cos v, -a\sin u \sin v, a\cos u) \times (-(b+a\cos u)\sin v, (b + a\cos u)\cos v, 0)|| \, du\, dv$
$= \int_0^{2\pi}\int_0^{2\pi} ||a(b + a\cos u)(-\cos u\cos v, -\cos u \sin v, -\sin u)|| \, du\, dv$ Since $b > a > 0$, $b + a\cos u > 0$ so the integral simplifies to

$$\int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\sqrt{\cos^2 u\cos^2 v + \cos^2 u \sin^2 v + \sin^2 u}\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\, du\, dv =$$

$$ab\left(\int_0^{2\pi}\int_0^{2\pi} du\, dv\right) + a^2\left(\int_0^{2\pi}\int_0^{2\pi} \cos u\, du\, dv\right) = 4ab\pi^2$$

Since $\int_0^{2\pi} \cos u\, du$ evaluates to 0 by inspection.

10. First, a parametric form for S is $\mathbf{g}(u, v) = (v\cos u, v^2, v \sin u)$ over $0 \le v \le 1$, $0 \le u \le 2\pi$. Thus $\left|\left|\frac{\partial \mathbf{g}}{\partial u} \times \frac{\partial \mathbf{g}}{\partial v}\right|\right| = ||(-v\sin u, 0, v\cos u) \times (\cos u, 2v, \sin u)|| = \left|\left|(-2v^2\cos u, v, -2v^2 \sin u)\right|\right| = \sqrt{4v^4 + v^2} = v\sqrt{4v^2+1}$ since $v \ge 0$.

(i) $\int\int x\, dS = \int_0^1\int_0^{2\pi} v^2\sqrt{v^4+1}\cos u\, du\, dv = \left(\int_0^1 v^2\sqrt{v^4+1}\, dv\right)\left(\int_0^{2\pi}\cos u\, du\right) = 0$ since $\int_0^{2\pi} \cos u\, du$ evaluates to 0.

(ii) $\int\int xz\, dS = \int_0^1\int_0^{2\pi} v^2\sqrt{v^4+1}\cos u \sin u\, du\, dv = \left(\int_0^1 v^2\sqrt{v^4+1}\, dv\right)\left(\int_0^{2\pi}\frac{\sin 2u}{2}\, du\right) = 0$ since $\int_0^{2\pi} \sin 2u\, du$ evaluates to 0.

(iii) $\displaystyle\int\int x^2\,dS = \int_0^1\int_0^{2\pi} v^3\sqrt{v^4+1}\cos^2 u\,du\,dv = \left(\int_0^1 v^3\sqrt{v^4+1}\,dv\right)\left(\int_0^{2\pi}\frac{1+\cos 2u}{2}\,du\right) =$
$\displaystyle\left[\frac{\sqrt{v^4+1}^3}{6}\right]_0^1\left(\int_0^{2\pi}\frac{1}{2}+\frac{\cos 2u}{2}\,du\right) = \frac{\sqrt[3]{2}-1}{6}\pi$ since $\displaystyle\int_0^{2\pi}\cos 2u\,du$ evaluates to 0.

(iv) $\displaystyle\int\int z^2\,dS = \int_0^1\int_0^{2\pi} v^3\sqrt{v^4+1}\sin^2 u\,du\,dv = \left(\int_0^1 v^3\sqrt{v^4+1}\,dv\right)\left(\int_0^{2\pi}\frac{1-\cos 2u}{2}\,du\right) =$
$\displaystyle\left[\frac{\sqrt{v^4+1}^3}{6}\right]_0^1\left(\int_0^{2\pi}\frac{1}{2}-\frac{\cos 2u}{2}\,du\right) = \frac{\sqrt[3]{2}-1}{6}\pi$ since $\displaystyle\int_0^{2\pi}\cos 2u\,du$ evaluates to 0.

16. Parametrize the sphere with $\mathbf{g} = (\sin u\cos v, \sin u\sin v, \cos u)$. for $0\le u\le\pi, 0\le v\le 2\pi$. Then $\mathbf{g}_u = (\cos u\cos v, \cos u\sin v, -\sin u), \mathbf{g}_v = (-\sin u\sin v, \sin u\cos v, 0)$. so $\dfrac{\partial\mathbf{g}}{\partial u}\times\dfrac{\partial\mathbf{g}}{\partial v} =$
$(\cos u\cos v, \cos u\sin v, -\sin u)\times(-\sin u\sin v, \sin u\cos v, 0) = (\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u)$

(i) $\displaystyle\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)du\,dv = \int_0^{2\pi}\int_0^{\pi}\cos^2 u\sin u\,du\,dv =$
$\displaystyle\left(\int_0^{2\pi}dv\right)\left(\int_0^{\pi}\cos^2 u\sin u\,du\right) = 2\pi\left[\frac{-\cos^3 u}{3}\right]_0^{\pi} = \frac{2\pi}{3}$

(ii) $\displaystyle\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)du\,dv = \int_0^{2\pi}\int_0^{\pi}\cos^3 u\sin u\,du\,dv =$
$\displaystyle\left(\int_0^{2\pi}dv\right)\left(\int_0^{\pi}\cos^3 u\sin u\,du\right) = 2\pi\left[\frac{-\cos^4 u}{4}\right]_0^{\pi} = 0$

(iii) $\displaystyle\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)du\,dv =$
$\displaystyle\int_0^{2\pi}\int_0^{\pi} -\sin^3 u\sin v\cos v + \sin^3 u\sin v\cos v\,du\,dv = 0$

18. $\displaystyle\left|\int\int_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS\right| = \left|\int\int_\Sigma \mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)du\,dv\right| \le \int\left|\int_\Sigma \mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)du\right|dv \le \int\int_\Sigma\left|\mathbf{F}\cdot\left(\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right)\right|du\,dv =$
$\displaystyle\int\int_\Sigma ||\mathbf{F}||\left|\left|\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right|\right||\cos\theta|\,du\,dv \le \int\int_\Sigma M\left|\left|\frac{\partial\mathbf{g}}{\partial u}\times\frac{\partial\mathbf{g}}{\partial v}\right|\right|du\,dv \le \int\int_\Sigma M\,dS = MS(\Sigma)$