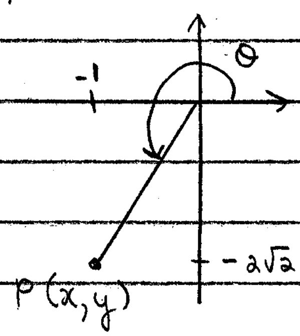
MATH 137 - FALL '03

Assignment 3 Solutions

Appendix D, #32



Since $\pi < \theta < \frac{3\pi}{2}$, θ is in the third quadrant and the coordinates of the point P are both negative. Since $\cos\theta = \frac{x}{r} = \frac{-1}{3}$ $\therefore$ $x = -1$, $r = 3$

and $y = -\sqrt{r^2 - x^2} = \sqrt{(3)^2 - (-1)^2} = -\sqrt{8} = -2\sqrt{2}$. Thus

$\sin\theta = \frac{y}{r} = \frac{-2\sqrt{2}}{3}$, $\tan\theta = \frac{y}{x} = \frac{-2\sqrt{2}}{-1} = 2\sqrt{2}$,

$\csc\theta = \frac{1}{\sin\theta} = \frac{-3}{2\sqrt{2}}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{-3}{1} = -3$ and

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{2\sqrt{2}}$.

Appendix D, #46 Prove $(\sin x + \cos x)^2 = 1 + \sin(2x)$.

We begin with the left side (L.S.) of the equation

$$
\begin{aligned}
(\sin x + \cos x)^2 &= \sin^2 x + 2\sin x\cos x + \cos^2 x \\
&= 1 + 2\sin x\cos x && \text{since } \sin^2 x + \cos^2 x = 1 \\
&= 1 + \sin(2x) && \text{since} \\
&= \text{R.S.} && \sin(2x) = 2\sin x\cos x \\
& && \text{by Identity 15a}
\end{aligned}
$$

which is what we wanted to prove.

Appendix D, #72

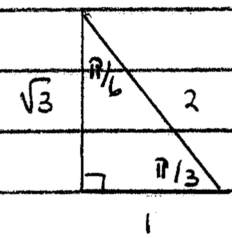
We wish to find all values of $x \in [0, 2\pi]$ that satisfy the equation $2 + \cos 2x = 3\cos x$. Since $\cos 2x = 2\cos^2 x - 1$ this is equivalent to solving

$$2 + 2\cos^2 x - 1 - 3\cos x = 0$$

or $$2\cos^2 x - 3\cos x + 1 = 0$$

or $$(2\cos x - 1)(\cos x - 1) = 0$$

$$\therefore \cos x = \frac{1}{2} \quad \text{or} \quad \cos x = 1$$



For $x \in [0, 2\pi]$ and $\cos x = \frac{1}{2}$ we obtain

$$x = \frac{\pi}{3} \quad \text{and} \quad x = \frac{5\pi}{3}$$

For $x \in [0, 2\pi]$ and $\cos x = 1$ we obtain

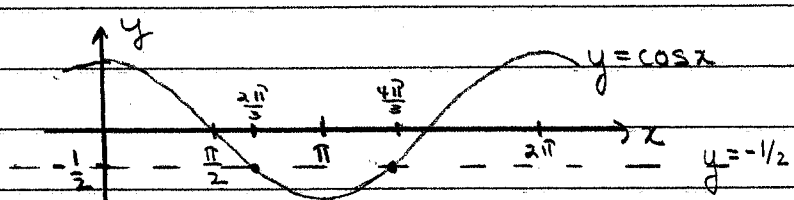
$$x = 0 \quad \text{and} \quad x = 2\pi$$

$\therefore$ the set of all required x is $\left\{0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi\right\}$

Appendix D, #74

We wish to find all values of $x \in [0, 2\pi]$ which satisfy $2\cos x + 1 > 0$ or equivalently

$$\cos x > -\frac{1}{2}$$

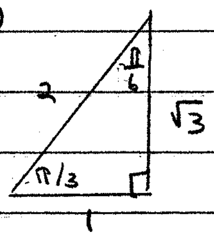


Now $\cos x = -1/2$ for $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$ if $x \in [0, 2\pi]$. From the graph of $\cos x$ it is easy to see that the values of x which satisfy both $x \in [0, 2\pi]$ and $\cos x > -\frac{1}{2}$ are

$$0 \le x < \frac{2\pi}{3} \quad \text{and} \quad \frac{4\pi}{3} < x \le 2\pi.$$

Section 1.6, #65

(a)



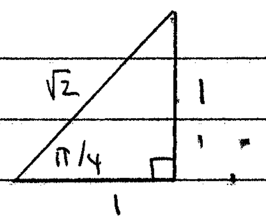
$\tan^{-1}(\sqrt{3}) = \arctan(\sqrt{3}) = \theta$

if and only if $\tan\theta = \sqrt{3}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Since $\tan(\pi/3) = \sqrt{3}$

$\therefore \theta = \tan^{-1}(\sqrt{3}) = \pi/3$

(b)



$\arcsin(-1/\sqrt{2}) = -\arcsin(1/\sqrt{2})$ since arcsin is an odd function

$\arcsin(1/\sqrt{2}) = \theta$ if and only if $\sin\theta = \frac{1}{\sqrt{2}}$ and $\theta \in [-\pi/2, \pi/2]$. Since $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \arcsin(1/\sqrt{2}) = \pi/4$ and thus $\arcsin(-1/\sqrt{2}) = -\pi/4$

Section 1.6, #69

Prove $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

Let $\theta = \sin^{-1}x$. Now $\theta = \sin^{-1}x$ if and only if $\sin\theta = x$ and $\theta \in [-\pi/2, \pi/2]$. If $\theta \in [-\pi/2, \pi/2]$ then $\cos\theta \geq 0$ so

$$
\begin{aligned}
\text{L.S.} = \cos(\sin^{-1}x) &= \cos\theta \\
&= \sqrt{1-\sin^2\theta} && \text{since } \cos^2\theta + \sin^2\theta = 1 \text{ and } \cos\theta \geq 0 \\
&= \sqrt{1-x^2} && \text{since } x = \sin\theta \\
&= \text{R.S.}
\end{aligned}
$$

as required.

Section 1.6, # 72

To evaluate $\sin(2\cos^{-1}x)$ we first let $\theta = \cos^{-1}x$ which implies $x = \cos\theta$ and $\theta \in [0, \pi]$. So

$$\begin{aligned} \sin(2\cos^{-1}x) &= \sin(2\theta) && \text{using the identity } \sin(2\theta) = 2\cos\theta \cdot \sin\theta \\ &= 2\cos\theta \cdot \sin\theta \\ &= 2x\sqrt{1-x^2} && \text{since } x = \cos\theta \text{ and } \sin\theta = \sqrt{1-\cos^2\theta} \text{ for } \theta \in [0,\pi] \end{aligned}$$

$\therefore \sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

Section 1.6, # 75

To find the domain and range of $g(x) = \sin^{-1}(3x+1)$ we first recall that the domain of $\sin^{-1}(x)$ is $[-1, 1]$. So the domain of g must be the set of all x values such that $-1 \le 3x+1 \le 1$ or $-2/3 \le x \le 0$.

$\therefore$ domain of $g = [-2/3, 0]$.

Since $g(x) = \sin^{-1}(3x+1)$ is obtained from $f(x) = \sin^{-1}(x)$ by streching and shifting the graph horizontally the range of g is the same as the range of f.

$\therefore$ range of $g = [-\pi/2, \pi/2]$

Sequence Notes Exercise #4

We want the first 8 terms of the sequence $x_n = \arctan(\sin(n\pi/2))$.

$$x_1 = \arctan(\sin(\pi/2)) = \arctan(1) = \pi/4$$
$$x_2 = \arctan(\sin(\pi)) = \arctan(0) = 0$$
$$x_3 = \arctan(\sin(3\pi/2)) = \arctan(-1) = -\arctan(1) = -\pi/4$$
$$x_4 = \arctan(\sin(2\pi)) = \arctan(0) = 0$$
$$x_5 = \arctan(\sin(5\pi/2)) = \arctan(1) = \pi/4$$
$$x_6 = \arctan(\sin(3\pi)) = \arctan(0) = 0$$
$$x_7 = \arctan(\sin(7\pi/2)) = \arctan(-1) = -\pi/4$$
$$x_8 = \arctan(\sin(4\pi)) = \arctan(0) = 0$$

Note: We observe the following pattern:

$$x_{2k} = 0$$
$$x_{4k-3} = \pi/4 \qquad \text{for } k \in \mathbb{Z}^+$$
$$x_{4k-1} = -\pi/4$$

Extra #1

$x = 0.5$: $\quad \arctan(x) = \arctan(0.5) \approx 0.4636$

$$\arcsin\left(\frac{x}{\sqrt{1+x^2}}\right) = \arcsin\left(\frac{0.5}{\sqrt{1.25}}\right) \approx 0.4636$$

$x = 3$: $\quad \arctan(3) \approx 1.2490$

$$\arcsin\left(\frac{3}{\sqrt{10}}\right) \approx 1.2490$$

$x = 10$: $\arctan(10) \approx 1.4711$

$\arcsin\left(\frac{10}{\sqrt{101}}\right) \approx 1.4711$

It would appear than $\arctan x = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$.

To prove this relationship we let $\theta = \arctan x$ which implies $x = \tan\theta$ and $\theta \in (-\pi/2, \pi/2)$. Now for $\theta \in (-\pi/2, \pi/2)$, $\sec\theta = \frac{1}{\cos\theta} > 0$. From the identity

$1 + \tan^2\theta = \sec^2\theta$ we obtain $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}$ since $\theta \in (-\pi/2, \pi/2)$ and $x = \tan\theta$.

$\therefore$ R.S. $= \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right) = \arcsin\left(\frac{\tan\theta}{\sec\theta}\right)$

$= \arcsin(\sin\theta)$

$= \theta$ since $\arcsin(\sin\theta) = \theta$ for all $\theta \in$ [redacted] $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= \arctan x$

$=$ L.S.

$\therefore$ $\arctan(x) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$ valid for all $x \in \mathbb{R}$.