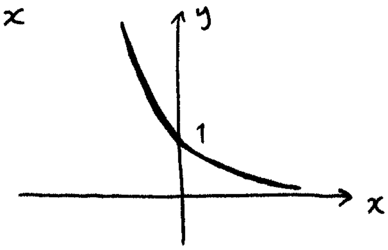
Solutions to Assignment #2 MATH 137

Section 1.5 #12

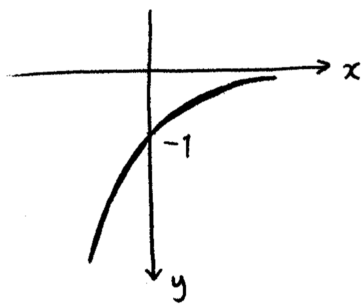
Start with $y = e^x$ and reflect along $x=0$ to obtain

$y = e^{-x}$



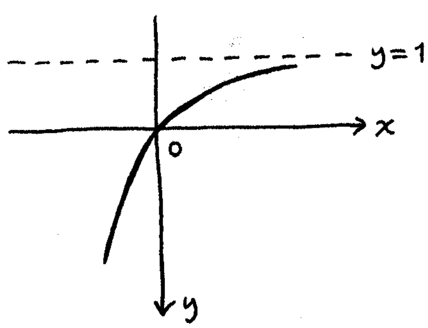
Next, reflect along $y=0$ to obtain

$y = -e^{-x}$



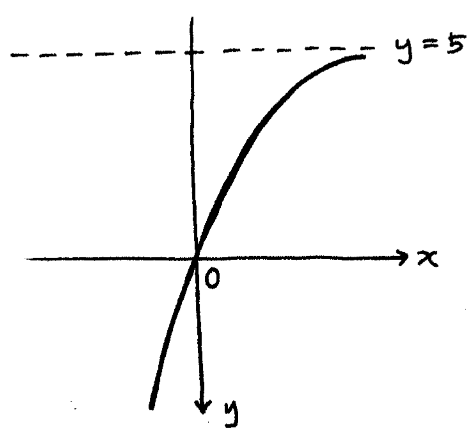
Next, shift graph 1 unit upward and obtain

$y = 1 - e^{-x}$



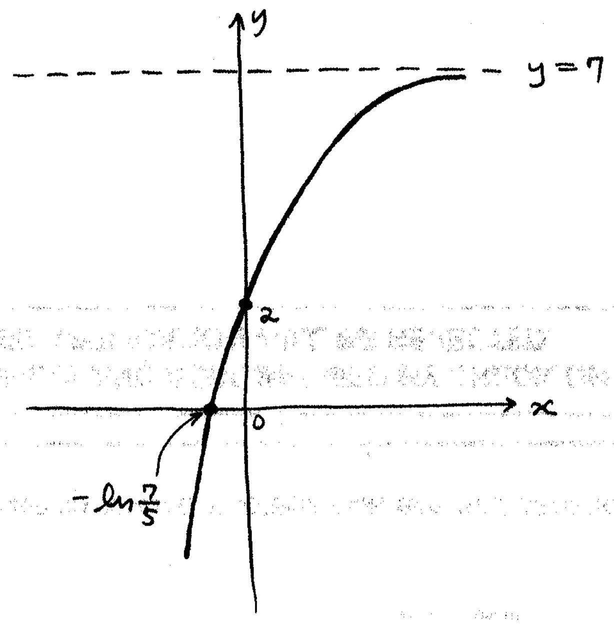
Next, vertically stretch by factor of 5, and obtain

$y = 5(1-e^{-x})$



Finally, shift 2 units upward and obtain

$y = 2 + 5(1 - e^{-x})$



$y = 0 \Rightarrow 0 = 2 + 5(1 - e^{-x})$

$$1 - e^{-x} = \frac{-2}{5}$$

$$e^{-x} = \frac{7}{5}$$

$$-x = \ln\frac{7}{5}$$

$$x = -\ln\frac{7}{5} \longleftarrow x\text{-intercept}$$

Section 1.6

\# 28. $y = \dfrac{1+e^x}{1-e^x} \Rightarrow y(1-e^x) = 1+e^x$

$\Rightarrow y - ye^x = 1+e^x \Rightarrow y-1 = (y+1)e^x$

$\Rightarrow e^x = \dfrac{y-1}{y+1} \Rightarrow \ln e^x = \ln\left(\dfrac{y-1}{y+1}\right)$

$\Rightarrow x = \ln\left(\dfrac{y-1}{y+1}\right)$

Finally, switch x and y to obtain

$$\boxed{y = \ln\left(\frac{x-1}{x+1}\right)}$$

\# 30. $y = \sqrt{x^2+2x}$ with condition $x > 0$.

$y^2 = x^2+2x \Rightarrow x^2+2x-y^2 = 0$

$\Rightarrow x = \dfrac{-2 \pm \sqrt{4-4(-y^2)}}{2} = -1 \pm \sqrt{1+y^2}$

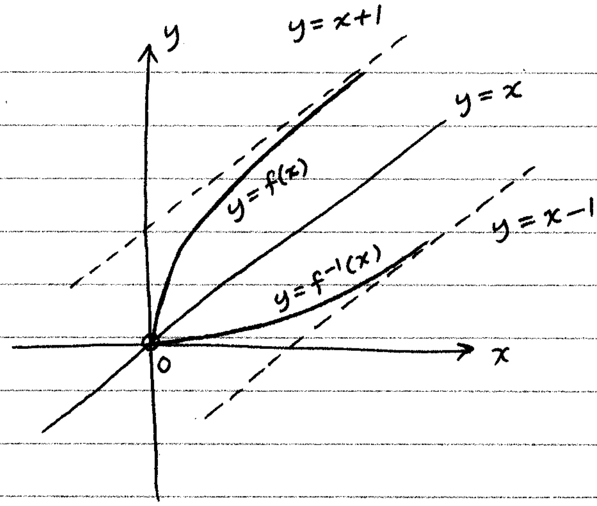
Note that $x = -1-\sqrt{1+y^2}$ is always negative, contradicting condition $x > 0$. Therefore

$x = -1+\sqrt{1+y^2}$

SWITCH x and y to obtain

$$\boxed{y = -1+\sqrt{1+x^2} = f^{-1}(x)}$$

Using a graphing calculator, we may draw



Extra Observations

① when $x > 0$, $x+1 = |x+1| = \sqrt{(x+1)^2} = \sqrt{x^2+2x+1}$
$> \sqrt{x^2+2x} = f(x)$

Hence $y = x+1$ lies above $y = f(x)$ always

② $f(x) - x = \sqrt{x^2+2x} - x = \dfrac{2x}{\sqrt{x^2+2x} + x}$

$$= \frac{2}{\frac{\sqrt{x^2+2x}}{\sqrt{x^2}} + 1} = \frac{2}{\sqrt{1+\frac{2}{x}} + 1} \qquad (x > 0)$$

Hence $\lim_{x\to\infty} \left(f(x) - x\right) = \lim_{x\to\infty} \dfrac{2}{\sqrt{1+\frac{2}{x}} + 1} = \dfrac{2}{\sqrt{1+0} + 1} = 1$

This means $f(x) - x$ approaches 1 as $x \to \infty$

i.e. $f(x)$ approaches $x+1$ as $x \to \infty$.

We have shown that $y = x+1$ is a "slant" asymptote for $y = f(x)$.

#52. (a) $\ln(\ln x) = 1 \Rightarrow e^{\ln(\ln x)} = e^1$

$\Rightarrow \ln x = e \Rightarrow e^{\ln x} = e^e$

$\Rightarrow \boxed{x = e^e}$

(b) $e^{ax} = C \cdot e^{bx} \Rightarrow \ln e^{ax} = \ln C + \ln e^{bx}$

$\Rightarrow ax = \ln C + bx$

$\Rightarrow (a-b)x = \ln C \Rightarrow \boxed{x = \dfrac{\ln C}{a-b}}$

#56. (a) Any x in the domain of f must satisfy

$2 + \ln x > 0$ and $x > 0$.

Now $2 + \ln x > 0 \Rightarrow \ln x > -2$

$\Rightarrow e^{\ln x} > e^{-2} \Rightarrow x > e^{-2}$

Hence the domain is given by

$x > e^{-2}$ and $x > 0$.

But $x > e^{-2}$ already implies $x > 0$, so the domain of f is given by just

$\boxed{x > e^{-2}}$

In the interval notation, this is (e^{-2}, ∞)

(b) $y = \ln(2+\ln x)$

$e^y = e^{\ln(2+\ln x)} = 2+\ln x$

$\Rightarrow \ln x = e^y - 2$

$\Rightarrow e^{\ln x} = e^{(e^y-2)} \Rightarrow x = e^{(e^y-2)}$

Switch x and y

$$\boxed{y = e^{(e^x-2)} = f^{-1}(x)}$$

Since e^x is defined for all $x \in \mathbb{R}$

the domain of f^{-1} is $\boxed{\mathbb{R}}$, the set of all real numbers.

Problems page 85

#11 Remember that $\log_a b = \dfrac{\ln b}{\ln a}$. Thus

$(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32)$

$$= \left(\frac{\ln 3}{\ln 2}\right)\left(\frac{\ln 4}{\ln 3}\right)\left(\frac{\ln 5}{\ln 4}\right)\cdots\left(\frac{\ln 32}{\ln 31}\right) = \frac{\ln 32}{\ln 2}$$

$$= \frac{\ln 2^5}{\ln 2} = \frac{5\ln 2}{\ln 2} = 5$$

For the last step, we can also proceed as

$$\frac{\ln 32}{\ln 2} = \log_2 32 = \log_2 2^5 = 5\cdot\log_2 2 = 5\cdot 1 = 5$$

#12 (a) We need to show that $f(-x) = -f(x)$.
From the formula, we compute

$$f(-x) = \ln\left(-x+\sqrt{(-x)^2+1}\right) = \ln\left(-x+\sqrt{x^2+1}\right)$$

This does not look like $-f(x)$, so we multiply by conjugate expression.

$$f(-x) = \ln\left(\left(-x+\sqrt{x^2+1}\right)\cdot\frac{-x-\sqrt{x^2+1}}{-x-\sqrt{x^2+1}}\right)$$

$$= \ln\left(\frac{(-x)^2-(x^2+1)}{-x-\sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-x-\sqrt{x^2+1}}\right)$$

$$= \ln\left(\frac{1}{x+\sqrt{x^2+1}}\right) = \ln\left(\left(x+\sqrt{x^2+1}\right)^{-1}\right)$$

$$= -1\cdot\ln\left(x+\sqrt{x^2+1}\right) = -1\cdot f(x) = -f(x)$$

We have shown that $f(-x) = -f(x)$ and thus $f(x)$ is an odd function.

(b) Set $y = \ln(x+\sqrt{x^2+1})$

We need to solve x in terms of y.

Apply exponential function to both sides so as to get rid of "$\ln$" on right side.

$$e^y = e^{\ln(x+\sqrt{x^2+1})} = x+\sqrt{x^2+1}$$

$$e^y = x+\sqrt{x^2+1}$$

$$e^y - x = \sqrt{x^2+1}$$

Square both sides to get rid of "$\sqrt{}$"

$$(e^y - x)^2 = x^2+1$$

$$e^{2y} - 2e^y \cdot x + x^2 = x^2+1$$

$$e^{2y} - 1 = 2e^y \cdot x$$

$$\Rightarrow \quad x = \frac{e^{2y}-1}{2e^y}$$

Finally switch variables x and y.

$$\boxed{y = \frac{e^{2x}-1}{2e^x} = f^{-1}(x)}$$

Note that we can simplify

$$f^{-1}(x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

13. Remember $\ln f(x)$ makes sense only when $f(x) > 0$. Thus we must have $x^2-2x-2>0$

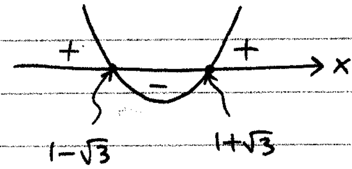
Now $x^2-2x-2=0 \iff x = \frac{2\pm\sqrt{4-4(-2)}}{2}$

$$= \frac{2\pm\sqrt{12}}{2} = \frac{2\pm2\sqrt{3}}{2}$$

$$= 1\pm\sqrt{3}$$

By looking at graph

$y = x^2 - 2x - 2$



we conclude that

$$x^2 - 2x - 2 > 0 \iff x < 1-\sqrt{3} \text{ or } x > 1+\sqrt{3}$$

Hence we need to solve $\ln(x^2-2x-2) \le 0$ under the assumption that $x < 1-\sqrt{3}$ or $x > 1+\sqrt{3}$.

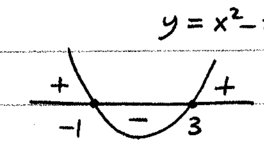
Now $\ln(x^2-2x-2) \le 0 \Rightarrow e^{\ln(x^2-2x-2)} \le e^0$

$$\Rightarrow x^2-2x-2 \le e^0 = 1$$

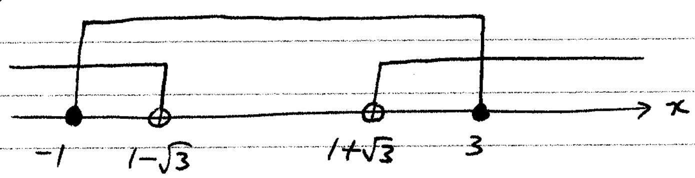
$$\Rightarrow x^2-2x-3 \le 0$$

$$\Rightarrow (x-3)(x+1) \le 0$$

$$\Rightarrow -1 \le x \le 3$$



Combining this with assumption $x < 1-\sqrt{3}$ or $x > 1+\sqrt{3}$



we conclude that the solution set is given by

$$\boxed{-1 \le x < 1-\sqrt{3} \quad \text{or} \quad 1+\sqrt{3} < x \le 3}$$

In the interval notation the solution set is

$$[-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$$