Note #6: One-Factor Copula Model

Wenhua Wang

Apr.28, 2006

Model description

Both Gaussian copula and double t copula are one-factor copula models. The difference is the distribution assumptions [2].

Suppose that we are interested in modeling the joint defaults of n different obligors. We define x_i $(1 \le i \le n)$:

$$x_i = a_i M + \sqrt{1-a_i^2} Z_i \qquad (1)$$

where M and the Z_i's have independent probability distributions with mean zero and standard deviation one. The variable x_i can be thought of as a default indicator variable for the i[th] obligor, i.e. the lower the value of x_i, the earlier a default is likely to occur. M is the same for all x_i while Z_i is an idiosyncratic component affecting only x_i.

Suppose that t_i is the time to default and Q_i is the cumulative probability distribution of t_i of the i[th] obligor. The copula model maps x_i to t_t on a "percentile to percentile" basis:

$$x = F_i^{-1}[Q_i(t)] \qquad (2)$$

or

$$t = Q_t^{-1}[F_i(x)]$$

where

F_i is the cumulative probability distribution for x_i.

Q_i is the cumulative probability distribution of t_i

F_i^{-1} is the inverse function of F_i

Q_i^{-1} is the inverse function of Q_i

The essence of the copula model is that we do not define the correlation structure between the variables of interest directly; instead we map the variables of interest into other more manageable variables and define a correlation structure between the latter.

From (1) and (2), we have:

$$Q_i(t \mid M) = \Pr ob(t_i < t \mid M) = H_i \left\{ \frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1-a_i^2}} \right\} \qquad (3)$$

In the Gaussian copula model, both *M* and the Z_i have standard normal distributions. In this case x_i also has a standard normal distribution, so that $H_i = F_i = N$ for all *i* where *N* is the cumulative normal distribution function.

In the double t-distribution copula model, both *M* and the Z_i have t-distributions. Because a standard t-distribution with *f* degrees of freedom has a mean of zero and a variance of *f/(f-2)*, so the random variable used for *M* in equation (1) is scaled by $\sqrt{(n_M - 2)/n_M}$ so that it has unit variance and the random variable used for the Z_i is scaled by $\sqrt{(n_Z - 2)/n_Z}$ for the same reason.

$$Q_i(t \mid M) = \Pr ob(t_i < t \mid M) = H_i\left\{\left(\frac{n_z}{n_z - 2}\right)^{1/2} \frac{F_i^{-1}[Q_i(t)] - r\left(\frac{n_M - 2}{n_M}\right)^{1/2} M}{\sqrt{1 - r^2}}\right\} \tag{4}$$

Equation (3) and (4) defines the cumulative probabilities of default by time t conditional on M. The variable M defines the default environment for the whole life of the model. Once M has been determined the cumulative probability of default Q_i is a known function of time. A one-factor copula model can be thought of as a model where there are many possible paths for the Q_i and the realization of *M* defines which will be taken for each *i*.

The definition for the hazard rate at time *t* conditional on *M* for company *i* is:

$$l_i(t \mid M) = \left.\frac{dQ_i(t \mid M)/dt}{1 - Q_i(t \mid M)}\right|_{t=t} \tag{5}$$

The equation (5) was referred as **a hazard rate path** by Hull[4].

The hazard rate at time t conditional on *M* can be calculated if the unconditional hazard rate is given, i.e. we can approximate $Q(t)$ by:

$$Q(t) = hr * t \tag{6}$$

where *hr* is the unconditional hazard rate per year.

Hence (5) can be calculated based on (3)/(4) and (6)

The cumulative t-distribution function is given by an incomplete beta function,

$$\int_{-\infty}^{t} f(u)\,du = \begin{cases} 1 - \frac{1}{2} I_x(\nu/2, 1/2) & \text{if} \quad t > 0, \\ \frac{1}{2} I_x(\nu/2, 1/2) & \text{otherwise,} \end{cases} \tag{7}$$

with

$$x = \frac{1}{1 + t^2/\nu}.$$

The parameter ? is called the number of degrees of freedom. The incomplete beta function is defined as

$$\mathrm{B}_x(a, b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t \qquad (8)$$

Algorithms

There are two critical problems are critical to the success of the project:

- The algorithm to calculate the inverse normal cumulative distribution function. The direct solution is to solve the equations numerically. This works but it slow. Fortunately, a good algorithm was found [5].
- A good algorithm to calculate inverse T-distribution cumulative function was not found, so numerical solution was used.

Source Code

Packatge opt.copula:

```
OneFactorCopula.java
GaussianCopula.java
DoubleTCopula.java
```

Package opt.test:

`CopulaTester.java` (run it by java opt.test.CopulaTester)

Package opt.util:

```
Bisection.java
FunctionI.java
Functions.java
IncompleteBeta.java
NDistribution.java
TDistribution.java
```

Test Results

Command:

java opt.test.CopulaTester > copula.csv

Input:

copula correlation rho (i.e. ai^2)=0.15

Unconditional hazard rate: 1% per year

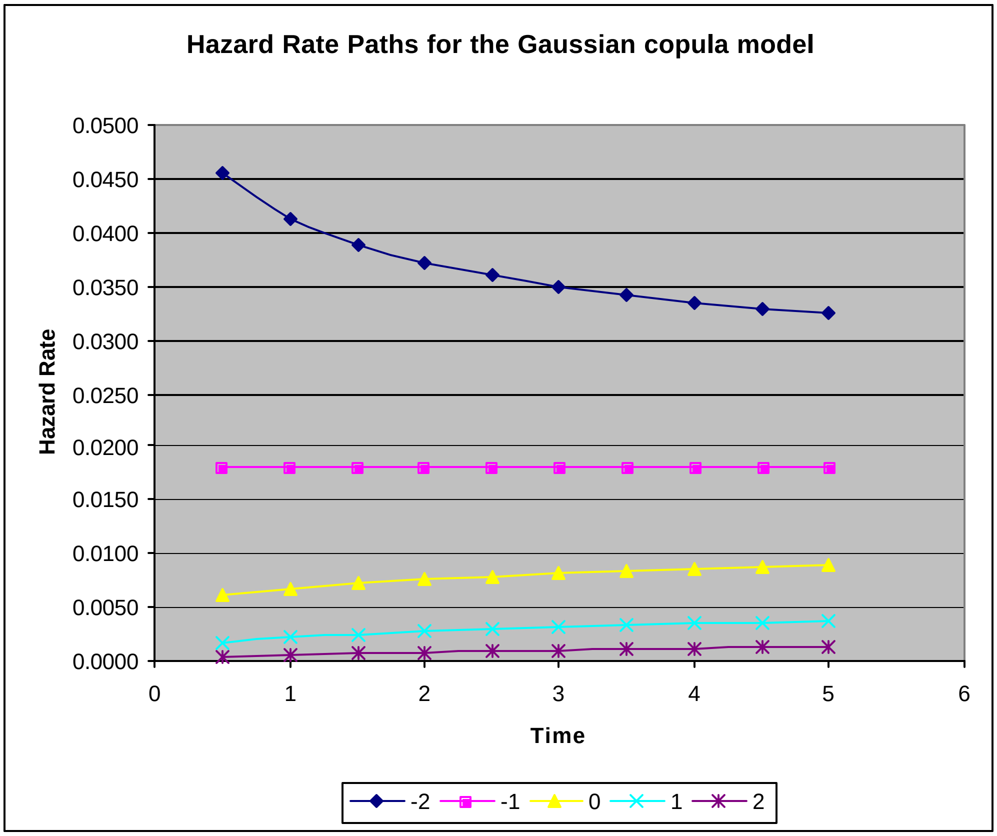
Output:

The program prints the results in CSV format to standard output. We can redirect it to a file e.g. copula.csv

The following is reformatted with Excel.

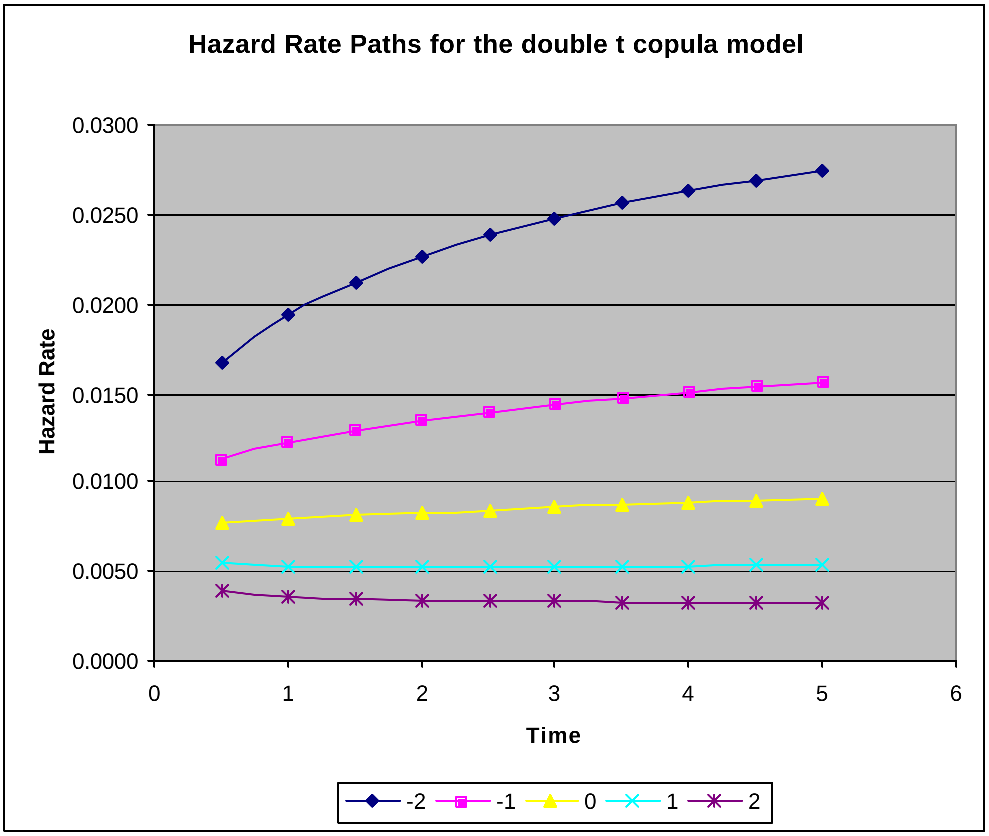
Gaussian copula model:

M/T	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
-2	0.0455	0.0413	0.0389	0.0372	0.0360	0.0350	0.0342	0.0335	0.0330	0.0325
-1	0.0180	0.0181	0.0181	0.0181	0.0181	0.0181	0.0181	0.0181	0.0181	0.0181
0	0.0061	0.0068	0.0072	0.0076	0.0079	0.0081	0.0083	0.0085	0.0087	0.0089
1	0.0017	0.0021	0.0024	0.0027	0.0029	0.0031	0.0033	0.0034	0.0036	0.0037
2	0.0004	0.0006	0.0007	0.0008	0.0009	0.0010	0.0011	0.0012	0.0013	0.0013



Double t copula model:

M/T	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
-2	0.0167	0.0194	0.0212	0.0227	0.0238	0.0248	0.0256	0.0263	0.0269	0.0275
-1	0.0112	0.0122	0.0129	0.0135	0.0140	0.0144	0.0147	0.0151	0.0154	0.0157
0	0.0077	0.0079	0.0081	0.0083	0.0084	0.0085	0.0087	0.0088	0.0089	0.0090
1	0.0054	0.0053	0.0052	0.0052	0.0052	0.0052	0.0052	0.0053	0.0053	0.0053
2	0.0039	0.0036	0.0035	0.0034	0.0033	0.0033	0.0033	0.0033	0.0033	0.0032



Conclusions

The Gaussian copula model shows that the future hazard rate decreases with the passage of time (except for a short initial period). This is not realistic.

Double t copula model is better than Gaussian. It shows that the hazard rate increases with the passage of time.

The testing results agree that of Hull and White[4].

References

[1] John Hull, Options, Futures, and Other Derivatives, 5/6th edition

[2] John Hull and Alan White, Valuation of a CDO and an nth to Default CDS without Monte Carlo Simulation, 2004

[3] David X. Li, On Default Correlation: A Copula Function Approach, 2000

[4] John Hull and Alan White, The Perfect Copula, 2005

[5] An algorithm for computing the inverse normal cumulative distribution function,
http://home.online.no/~pjacklam/notes/invnorm/

[6] http://en.wikipedia.org/wiki/Student's_t-distribution

[7] Halley's Rational Formula for square roots
http://www.mathpath.org/Algor/squareroot/algor.square.root.halley.htm

[8] Peter John Acklam, A small paper on Halley's method,
http://home.online.no/~pjacklam/notes/halley/halley.pdf