

Perfect Pyramids

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Abstract

This paper discusses rational edged tetrahedra, in 3, 4 and n dimensions, with rational volume. The main results are (i) a proof of the existence of infinitely many tetrahedra with rational edge-lengths, face-areas and volume and (ii) a proof that there exist dimensions for which all regular hypertetrahedra with rational edge-lengths have rational hypervolume.

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1 Introduction

Define a perfect pyramid to be a tetrahedron with integer edge-lengths, face areas and volume. This definition can be extended to higher and lower dimensions. For example in one dimension any line segment of integer length is a perfect 1-pyramid.

More interestingly in two dimensions any triangle with integer edges and area, such as the (3, 4, 5) triangle, is a perfect 2-pyramid. Hero's formula for the area, Δ , of a triangle in terms of the edge-lengths is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}.$$

This can be rewritten in matrix form as follows:

$$(4\Delta)^2 = \begin{bmatrix} a^2 & b^2 & c^2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}.$$

The complete integer solutions to this diophantine equation [Dic52] have been known since Euler first found them. Carmichael's parametric version [Car52] is

$$\begin{aligned} a &= n(m^2 + k^2) \\ b &= m(n^2 + k^2) \\ c &= (m+n)(mn - k^2) \\ \Delta &= kmn(m+n)(mn - k^2) \end{aligned}$$

which produces one member of each similarity class of Heron triangles for any integers m , n and k such that $\gcd(m, n, k) = 1$, $mn > k^2 > \frac{m^2 n}{2m+n}$ and $m \geq n \geq 1$.

2 Volume Of A Tetrahedron

Suppose one were to ask for the analogous formula for the volume of a tetrahedron. Consider the tetrahedron in Figure 1. Position the axes so that the coordinates of the four vertices are given by $A : (x_A, y_A, 0)$, $B : (0, 0, 0)$, $C : (a, 0, 0)$ and $D : (x_D, y_D, z_D)$. Note that the coordinates of A can be expressed via the

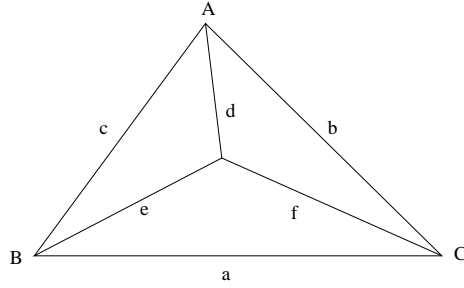


Figure 1:

cosine rule and area formula as

$$x_A = \frac{a^2 + c^2 - b^2}{2a} \quad \text{and} \quad y_A = \frac{2\Delta_{abc}}{a}.$$

The volume of a tetrahedron is given by

$$V = \frac{1}{3} z_D \Delta_{abc}.$$

Since the base, Δ_{abc} , can be expressed in terms of the edge lengths via Heron's formula one need only express the height, z_D , in terms of the edge lengths. The coordinates of vertex D can be obtained by intersecting three spheres with centres at A , B and C having radii d , e and f respectively. The surfaces of these three spheres are defined by

$$\begin{aligned} x^2 + y^2 + z^2 &= e^2 \\ (x - a)^2 + y^2 + z^2 &= f^2 \\ (x - x_A)^2 + (y - y_A)^2 + z^2 &= d^2 \end{aligned}$$

and their mutual intersection occurs at $(x, y, z) = (x_D, y_D, \pm z_D)$. Thus

$$\begin{aligned}x_D &= \frac{a^2 + e^2 - f^2}{2a} \\y_D &= \frac{c^2 + e^2 - d^2 - 2x_D x_A}{2y_A} \\z_D^2 &= e^2 - x_D^2 - y_D^2.\end{aligned}$$

Substituting the expressions for x_A , y_A , x_D and y_D into the last equation one obtains

$$z_D^2 = \frac{16^2 \Delta_{aef}^2 \Delta_{abc}^2 - [(e^2 - f^2 - a^2)(a^2 + b^2 - c^2) + 2a^2(b^2 + f^2 - d^2)]^2}{64 \Delta_{abc}^2 a^2}.$$

Finally recalling that $9V^2 = z_D^2 \Delta_{abc}^2$ leads to

$$(24aV)^2 = 16^2 \Delta_{aef}^2 \Delta_{abc}^2 - [(e^2 - f^2 - a^2)(a^2 + b^2 - c^2) + 2a^2(b^2 + f^2 - d^2)]^2.$$

Expanding this gives

$$\begin{aligned}(12V)^2 &= (a^2 + d^2)(-a^2 d^2 + b^2 e^2 + c^2 f^2) + (b^2 + e^2)(a^2 d^2 - b^2 e^2 + c^2 f^2) \\&\quad + (c^2 + f^2)(a^2 d^2 + b^2 e^2 - c^2 f^2) - a^2 b^2 c^2 - a^2 e^2 f^2 - b^2 d^2 f^2 - c^2 d^2 e^2.\end{aligned}$$

So the volume of a tetrahedron is $(12V)^2 = X^t M X$ where $X = [a^2 b^2 c^2 d^2 e^2 f^2]^t$ and

$$M = \begin{bmatrix} -d^2 & d^2 & d^2 + b^2 & 0 & 0 & 0 \\ e^2 - c^2 & -e^2 & e^2 & 0 & 0 & 0 \\ f^2 & f^2 - a^2 & -f^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2 & a^2 & a^2 - b^2 \\ 0 & 0 & 0 & b^2 - c^2 & -b^2 & b^2 \\ 0 & 0 & 0 & c^2 & c^2 - a^2 & -c^2 \end{bmatrix}$$

3 Integer Volume Tetrahedra

Using these equations one can search for tetrahedra with integer volumes or even perfect pyramids. Recall that triangles can be categorised into equilateral, isosceles and scalene. Similarly, by equating various numbers of edges a tetrahedron becomes more symmetric and the corresponding problem becomes (one hopes) a little easier to solve.

We now consider each case of Table (1) in detail.

• 1(i). Substituting the equalities $a = b = c = d = e = f$ into the expression for the volume of a tetrahedron leads to

$$(12V)^2 = 2a^6,$$

One parameter tetrahedron	# such that $V \in \mathbb{N}$	example
(i) $a = b = c = d = e = f$	0	-
Two parameter tetrahedra		
(i) $a = b = c = d = e, f$	0	-
(ii) $a = b = c = d, e = f$	0	-
(iii) $a = c = d = f, b = e$	∞	$(a, b) = (9, 12)$
(iv) $a = b = c, d = e = f$	0	-
(v) $a = d = f, b = c = e$	0	-
Three parameter tetrahedra		
(i) $a = b = c = d, e, f$?	?
(ii) $a = c = d = f, b, e$	∞	$(a, b, e) = (7, 4, 6)$
(iii) $a = b = c, d = e, f$?	?
(iv) $a = f = d, b = c, e$?	?
(v) $a = f = d, b = e, c$	finite ?	$(a, b, c) = (11, 15, 16)$
(vi) $a = d, b = e, c = f$	∞	$(a, b, c) = (21, 20, 11)$
(vii) $a = e, b = f, c = d$?	?
(viii) $a = b, c, d = e = f$	∞	$(a, c, d) = (12, 8, 9)$
(ix) $a = d, b = f, c = e$	∞	$(a, b, c) = (12, 7, 11)$
(x) $a = e, b = c, d = f$?	?

Table 1: Symmetrised tetrahedra

$$V = \frac{\sqrt{2}a^3}{12}.$$

Clearly if $a \in \mathbb{N}$ then $V \notin \mathbb{N}$. Hence there are no solutions in this case.

- 2(i). Substitution gives

$$(12V)^2 = a^2 f^2 (3a^2 - f^2).$$

Without loss of generality let $\gcd(a, f) = 1$ and $a, f, V \in \mathbb{N}$; then there exists an integer, p say, such that

$$3a^2 - f^2 = p^2$$

$$p^2 + f^2 = 3a^2$$

$$p^2 + f^2 \equiv 0 \pmod{3}.$$

This implies that $3|p$ and $3|f$ and hence $3|a$, which contradicts the assumption of relative primality. Thus no perfect pyramids of this form exist.

- 2(ii). Substitution gives

$$(12V)^2 = a^2 (6a^2 e^2 - (a^2 + e^2)^2).$$

Without loss of generality let $\gcd(a, e) = 1$ and $a, e, V \in \mathbb{N}$, then there exists an integer, p say, such that

$$6a^2 e^2 - (a^2 + e^2)^2 = p^2,$$

$$p^2 + (a^2 + e^2)^2 \equiv 0 \pmod{3}.$$

This implies that $3|p$ and $3|(a^2 + e^2)$; hence $3|a$ and $3|e$, clearly a contradiction again, leading to no perfect pyramids of this form.

- 2(iii). This time

$$(12V)^2 = b^4(4a^2 - 2b^2).$$

If $\gcd(a, b) = 1$ and $a, b, V \in \mathbb{N}$, then there exists an integer, p say, such that

$$4a^2 - 2b^2 = p^2.$$

Clearly p must be even so letting $p = 2P$ gives

$$2a^2 - b^2 = 2P^2$$

which means that b is even. Hence $b = 2B$ say leads to

$$a^2 - P^2 = 2B^2.$$

Now a and P must have the same parity so let $a = Q + S$, $P = Q - S$ to give

$$2QS = B^2.$$

As before B is even so $B = 2D$, say, leads to

$$QS = 2D^2.$$

Without loss of generality let $Q = 2R$ where $\gcd(R, S) = 1$ so that

$$RS = D^2$$

which has solutions of the form $(R, S, D) = (r^2, s^2, rs)$ for some integers r and s . Thus back substitution gives an infinite family of integer sided tetrahedra with rational volume namely

$$\begin{aligned} a &= 2r^2 + s^2 \\ b &= 4rs \\ V &= \frac{8}{3}r^2s^2(2r^2 - s^2). \end{aligned}$$

- 2(iv). Substitution gives $(12V)^2 = a^4(3d^2 - a^2)$ which is the same form as 2(i) and so no solutions exist.

- 2(v). Substitution gives

$$(12V)^2 = (a^2 + b^2)(3a^2b^2 - a^4 - b^4).$$

If $\gcd(a, b) = 1$ and $a, b, V \in \mathbb{N}$ then there exist integers, m and n say, such that

$$a^2 + b^2 = m^2 \quad \text{and} \quad 3a^2b^2 - a^4 - b^4 = n^2.$$

The only solutions to the latter (an elliptic curve of rank 0) occur when $a^2 = b^2$ which makes the former equation insoluble in integers.

- 3(i). Substitution gives $(12V)^2 = a^2(f^2(2a^2 + e^2 - f^2) - (a^2 - e^2)^2)$. It is unknown whether or not this has solutions but none exist in the range $e, f \leq 2a$, where $a \leq 100$.

- 3(ii). Substitution gives $(12V)^2 = b^2e^2(4a^2 - (b^2 + e^2))$. If $\gcd(a, b, e) = 1$ and $a, b, e, V \in \mathbb{N}$ then there exists an integer, n say, such that

$$4a^2 - (b^2 + e^2) = n^2.$$

Rearranging leads to

$$b^2 + e^2 + n^2 = 4a^2.$$

This means that b, e and n must be even numbers so letting $b = 2B, e = 2E, n = 2N$ leads to

$$B^2 + E^2 + N^2 = a^2.$$

Now $\gcd(a, b, e) = 1$ implies that a must be odd. Any number with all its prime divisors equivalent to one modulo 4 has a representation as a sum of two squares. Furthermore it turns out that only numbers of the form $4^s(8m + 7)$ can be represented as a sum of four squares (and no fewer). But $a^2 \equiv 7 \pmod{8}$ is impossible and a is odd and hence a^2 never takes the form $4^s(8m + 7)$. So finally the above equation is soluble for all integers, a , in the following set.

$A = \{a : a \text{ is odd and there exists at least one prime } p \text{ such that } p|a \text{ implies } p \equiv 3 \pmod{4}\}$.

For example $a = 3, 7, 9, 11, 15, \dots$ provide the solutions

$$1^2 + 2^2 + 2^2 = 3^2,$$

$$2^2 + 3^2 + 6^2 = 7^2,$$

$$1^2 + 4^2 + 8^2 = 9^2,$$

$$2^2 + 6^2 + 9^2 = 11^2,$$

$$2^2 + 5^2 + 14^2 = 15^2.$$

Since the set $\{p : p \text{ is prime, } p \equiv 1 \pmod{4}\}$ is infinite in size so is the set obtained from it $\{a = 3p : p \text{ is prime, } p \equiv 1 \pmod{4}\}$ which is a subset of the set A . Thus $|A| = \infty$ and there are an infinite number of solutions to the last equation. Note that from the second example one obtains $(a, b, e) = (7, 4, 6)$ which has a volume of 24.

- 3(iii). Substitution gives $(12V)^2 = a^2(16\Delta_{adf}^2 - a^2f^2)$ which has no known solutions for $a \leq 2d, f \leq 3d, d \leq 100$.

- 3(iv). Substitution gives $(12V)^2 = b^2e^2(3a^2 - e^2) - a^2(a^2 - b^2)^2$ which has no known solutions for $b \leq 2a, e \leq 2a, a \leq 100$.

- 3(v). Substitution gives $(12V)^2 = b^4(3a^2 - 2b^2 + c^2) - a^2(a^2 - c^2)^2$ which has the solutions $(a, b, c) = (11, 15, 16), (12, 10, 15), (16, 10, 15), (20, 26, 39)$ for $b \leq 2a, c \leq 3a, a \leq 20$.

- 3(vi). Substitution gives

$$(12V)^2 = 2(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)$$

which has 4 known solutions for $a \leq 156$, $b \leq a$, $c \leq a$. These are

$$(a, b, c) = (21, 20, 11), (72, 65, 33), (100, 91, 69), (100, 99, 21).$$

It will be shown in the next section that each of these generates an infinite set of solutions.

- 3(vii). Substitution gives

$$(12V)^2 = 5a^2b^2c^2 - a^4b^2 - b^4c^2 - c^4a^2$$

which has no solutions for $a \leq 100$, $b \leq a$, $c \leq a$.

- 3(viii). Substitution gives

$$(12V)^2 = c^2(4a^2d^2 - c^2d^2 - a^4)$$

which has one solution for $a \leq 2d$, $c \leq 2d$, $d \leq 10$, namely $(a, c, d) = (12, 8, 9)$. From this one can generate an infinite set of solutions.

Let $a = \frac{4}{3}d$ to give

$$Q^2 = 320d^2 - 9^2c^2$$

hence

$$Q^2 + K^2 = 5D^2$$

where $K = 9c$, $D = 8d$, $Q = \frac{108V}{cd}$. This equation has a parametric solution of

$$\begin{aligned} Q &= 2(5q^2 - p^2) \\ K &= p^2 - 10pq + 5q^2 \\ D &= p^2 - 2pq + 5q^2. \end{aligned}$$

Whence a set of integer volume pyramids is given by

$$\begin{aligned} ga &= 12(p^2 - 2pq + 5q^2) \\ gc &= 8(p^2 - 10pq + 5q^2) \\ gd &= 9(p^2 - 2pq + 5q^2) \end{aligned}$$

and g is the gcd of the three right hand sides.

• 3(ix). Substitution gives $(12V)^2 = a^2(16\Delta_{abc}^2 - a^4)$ which has three solutions for $a \leq 2c$, $b \leq 3c$, $c \leq 35$, namely $(a, b, c) = (12, 7, 11), (28, 15, 27), (36, 19, 35)$ from which an infinite set of solutions can be obtained. Rewriting the area using Heron's formula gives

$$\begin{aligned} (12V)^2 &= 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - 2a^4 - b^4 - c^4, \\ &= 2a^2(b^2 + c^2 - a^2) - (b - c)^2(b + c)^2. \end{aligned}$$

So let $\frac{c-b}{a} = \frac{11-7}{12} = \frac{1}{3}$ or $a = 3(c-b)$ and $Q = \frac{12V}{c-b}$ to give

$$\begin{aligned} Q^2 &= 18(b^2 + c^2 - 9(b-c)^2) - (b+c)^2, \\ &= 322bc - 145b^2 - 145c^2, \\ &= (63b - 39c)^2 - 34(11b - 7c)^2. \end{aligned}$$

Letting $B = 63b - 39c$ and $C = 11b - 7c$ leads to the equation

$$Q^2 + 34C^2 = B^2$$

which has the solutions

$$\begin{aligned} Q &= p^2 - 34q^2 \\ C &= 2pq \\ B &= p^2 + 34q^2. \end{aligned}$$

Substituting this parametrisation into the appropriate equations above and solving for a , b and c leads to

$$\begin{aligned} ga &= 12p^2 - 144pq + 408q^2 \\ gc &= 7p^2 - 78pq + 238q^2 \\ gd &= 11p^2 - 126pq + 374q^2 \end{aligned}$$

where g is the gcd of the three right hand sides.

• 3(x). Substitution gives $(12V)^2 = 5a^2b^2d^2 - a^2b^4 - a^4d^2 - b^2d^4$ which has no solutions for $a \leq 3d$, $b \leq 2d$, $d \leq 100$.

4 Perfect Pyramids

Having thus far found many different (though by no means all) sets of tetrahedra with integer volume one can return to the problem of searching for perfect pyramids.

A brute force search is very slow to uncover any — in fact the tetrahedron

$$(a, b, c, d, e, f) = (117, 80, 53, 52, 51, 84)$$

is the only perfect pyramid with $a, b, c, d, e, f \leq 156$. The volume is 18144 while the face areas are $\Delta_{abc} = 1800$, $\Delta_{aef} = 1890$, $\Delta_{bdf} = 2016$, $\Delta_{cde} = 1170$.

Next, one can consider the restrictions to one, two and three parameter tetrahedra. Since no one-parameter tetrahedron can have integer volume there can be no perfect pyramids of this form.

Similarly, for two-parameter tetrahedra one need only consider case (iii). Here each face is a triangle with sides $(a, a, b) = (2r^2 + s^2, 2r^2 + s^2, 4rs)$. Thus the length, h , of the altitude to the side b is given by Pythagorus' theorem as

$$\begin{aligned} h^2 &= a^2 - (b/2)^2 \\ &= (2r^2 + s^2)^2 - (2rs)^2 \\ &= 4r^4 + s^4 \end{aligned}$$

which has no solutions in positive integers.

A more profitable search can be made by considering the three-parameter tetrahedra. For cases (i) and (iii) the bases are equilateral triangles which cannot have integer area. This immediately precludes the possibility of perfect pyramids of this form.

However, consider case (vi). If the base (a, b, c) is a Heron triangle then all four face areas are integral (since they are all identical). Using Carmichael's parametrisation for Heron triangles one can quickly discover the five perfect pyramids of Table (2). Note that the third example here has been discovered previously [Guy87] as a solution to problem D22 in R. Guy's Unsolved Problems in Number Theory [Guy81].

m	n	k	a	b	c	V
21	20	15	888	875	533	37608480
39	34	26	2873	2748	1825	1355172000
39	35	25	203	195	148	611520
58	33	30	1804	1479	1183	214582368
77	68	44	2431	2296	2175	1403038560

Table 2: Perfect Pyramids of form (a, b, c, a, b, c)

Theorem 1 *There exist an infinite number of perfect pyramids with edges of the form (a, b, c, a, b, c) .*

Proof : First recall that the volume of a tetrahedron in this case is given by

$$(12V)^2 = 2(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2). \quad (1)$$

Now replacing a, b and c via Carmichael's parametrisation first gives

$$\begin{aligned} a^2 + b^2 - c^2 &= 2mn[k^2(m+n)^2 - (mn - k^2)^2], \\ a^2 + c^2 - b^2 &= 2n(m+n)(mn - k^2)(m^2 - k^2), \\ b^2 + c^2 - a^2 &= 2m(m+n)(mn - k^2)(n^2 - k^2), \end{aligned}$$

and hence one obtains

$$(12V)^2 = [4mn(m+n)(mn - k^2)]^2 [k^2(m+n)^2 - (mn - k^2)^2] (m^2 - k^2)(n^2 - k^2).$$

So there exists an integer, v say, such that

$$v^2 = [k^2(m+n)^2 - (mn - k^2)^2] (m^2 - k^2)(n^2 - k^2).$$

Now dividing through by k^8 and setting $Z = v/k^4$, $x = m/k$, $y = n/k$ leads to

$$Z^2 = (x + y - xy + 1)(x + y + xy - 1)(x - 1)(x + 1)(y - 1)(y + 1).$$

Notice that this is a quartic in x and y and it has rational solutions corresponding to the perfect pyramids in Table (2), one of which is $(x, y, Z) = (7/5, 4/3, 56/25)$.

The aim now is to show that this solution generates an infinite number of solutions by transforming the equation to an elliptic curve of rank ≥ 1 .

Let $y = 4/3$ and $Z = z/9$ to give

$$z^2 = 7(-x + 7)(7x + 1)(x - 1)(x + 1) \quad (2)$$

which has a rational point $(x_1, z_1) = (7/5, 504/25)$. Before transforming this it is first useful to obtain a second rational point on the quartic curve by a method analogous to the tangent-chord process.

Note that the parabola $z_p = a_p x^2 + b_p x + c_p$ where $a_p = -13057/(2^2 \cdot 3^4 \cdot 7)$, $b_p = 40705/(2 \cdot 3^4 \cdot 5)$, $c_p = -12607/(2^2 \cdot 3^4)$ meets the quartic curve (2) at the point $(x_1, z_1) = (7/5, 504/25)$ such that $z'_p = z'$ and $z''_p = z''$. Hence intersecting this quadratic with equation (2) leads to another quartic equation, $g(x)$ say, which has, by its construction, a root of multiplicity 3 at $x = 7/5$ and a second root which corresponds to a second rational point on equation (2) see Figure 2.

Thus equating coefficients of powers of x in

$$g(x) = z^2 - z_p^2 = (x - 7/5)^3 (P_2 x - Q_2) = 0$$

leads to $P_2 = (-5^3 \cdot 677 \cdot 4993)/(2^4 \cdot 3^8 \cdot 7^2)$, $Q_2 = (-5^3 \cdot 3348577)/(2^4 \cdot 3^8 \cdot 7)$ and so the second rational point on (2) is (x_2, z_2) where

$$x_2 = \frac{Q_2}{P_2} = \frac{7 \cdot 3348577}{677 \cdot 4993}, z_2 = \frac{2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 23 \cdot 89 \cdot 587 \cdot 3167}{677^2 \cdot 4993^2}.$$

Transform the quartic curve (2) so that the leading term becomes a positive

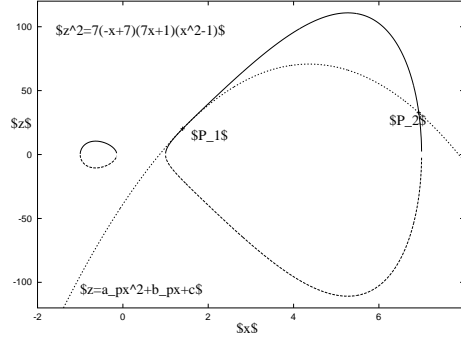


Figure 2:

rational square by setting $x = (7X + 1)/(5X)$ and $z = W/(5^2 X^2)$. Hence equation (2) becomes

$$W^2 = 7(28X - 1)(54X + 7)(12X + 1)(2X + 1). \quad (3)$$

Notice that the rational point at $(x_1, z_1) = (7/5, 504/25)$ has been transformed to $X = \infty$ which is why the second rational point was required. Now equation

(3) can be transformed into a monic quartic equation lacking the cubic term by setting

$$X = \frac{u - 256}{2^3 \cdot 3^3 \cdot 7}, \quad W = \frac{w}{2^3 \cdot 3^4 \cdot 7}.$$

Hence

$$w^2 = (u - 310)(u - 60)(u - 130)(u + 500)$$

or

$$w^2 = u^4 - 6 \cdot 30550u^2 + 4 \cdot 7733000u - 1209000000. \quad (4)$$

Applying Mordell's transformation, namely, $2u = (t + q)/(s - p)$ and $w = 2s - u^2 + p$ where $p = 30550$, $q = -7733000$ and $r = -1209000000$ leads to the elliptic curve

$$t^2 = 4s^3 - g_2s + g_3 \quad (5)$$

where $g_2 = 3p^2 + r = 1590907500$ and $g_3 = q^2 + pr - p^3 = -5648052375000$. Now since four divides the right hand side of equation (5) we conclude that t must be even, so letting $t = 2T$ leads to

$$T^2 = s^3 - 397726875s - 2 \cdot 5^3 \cdot 5648052375$$

that is

$$T^2 = (s + 17850)(s + 3675)(s - 21525). \quad (6)$$

The rational point (x_2, z_2) on equation (2) can be transformed into a rational point on equation (6), namely (s_1, T_1) where

$$s_1 = \frac{5^2 \cdot 13 \cdot 416834008069}{2^6 \cdot 11^2 \cdot 19^2 \cdot 37^2} \quad \text{and} \quad T_1 = \frac{78688321296560732005909000}{(2^6 \cdot 11^2 \cdot 19^2 \cdot 37^2)^2}.$$

Since s_1 and T_1 are not integers then by a theorem of Lutz and Nagel (see [Sil86] p. 221), the point (s_1, T_1) must have infinite order. Thus equation (6) has an infinite number of rational solutions which, in turn, implies that equation (2) has an infinite number of rational solutions.

The final hurdle is the transformation of the triangle inequalities since only solutions satisfying $mn > k^2 > (m^2n)/(2m + n)$ and $m \geq n \geq 1$ lead to valid perfect pyramids in equation (1). Now

$$\begin{aligned} mn > k^2 &\longrightarrow xy > 1 &\longrightarrow x > \frac{3}{4} \\ m > n &\longrightarrow x > y &\longrightarrow x > \frac{4}{3} \\ k^2 > \frac{m^2n}{2m+n} &\longrightarrow 1 > \frac{x^2y}{2x+y} &\longrightarrow -\frac{1}{2} < x < 2. \end{aligned}$$

So the rational points (x, z) on equation (2) which correspond to perfect pyramids are those for which $4/3 < x < 2$ and $(7\sqrt{17 \cdot 31})/9 < z < 15\sqrt{17}$. (In particular the second rational point, (x_2, z_2) , found above does not lie in this range and so does not produce a new perfect pyramid.) Continuing the transformation of the triangle inequalities leads ultimately to the corresponding

inequality $s > 46027 + 4900\sqrt{17 \cdot 31} > 127963$ for rational points on the elliptic curve (6). Since equation (6) has an infinite number of rational points (s, T) satisfying $s > r$ for any constant r , one concludes that equation (1) has an infinite number of solutions corresponding to perfect pyramids. ■

Notice that this also provides an infinite number of solutions for case 3(vi) in section 3.

Searches for perfect pyramids of the form 3(v), (viii) and (ix) have so far failed to turn up any examples.

5 Integer Volume Simplex

For a four dimensional tetrahedron, often called a simplex, (see figure (3)) one can use a similar technique to express the hypervolume, \mathcal{V} , in terms of the edge lengths.

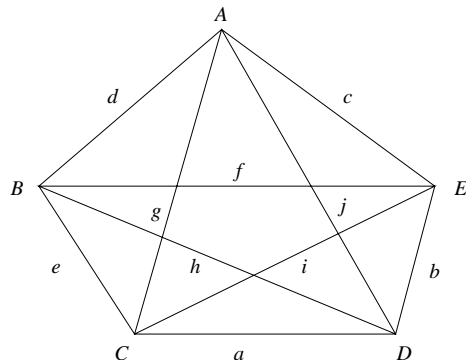


Figure 3:

First note that

$$\mathcal{V}_{abcde} = \frac{1}{4} \mathcal{V}_{abcd} w_E$$

where w_E is the perpendicular height of the point E above the tetrahedron $(ABCD)$. The volume of the base can be expressed in terms of the edge lengths using the expression in section 2. So proceeding as before one need only find w_E in terms of the edge lengths. Intersecting the four hyperspheres with centres at A, B, C, D and radii c, f, i, b respectively one obtains the hypervolume of a simplex, namely,

$$(96\mathcal{V})^2 = X^t M X$$

where $X = [a^2 b^2 c^2 d^2 e^2 f^2 g^2 h^2 i^2 j^2]^t$, $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ and

$P =$

$$S = \begin{bmatrix} -d^4 - 16\Delta_{cdf}^2 & 2d^2(f^2 + g^2) & -2(b^2g^2 + i^2j^2) & 2j^2(i^2 + f^2) & 2(d^2b^2 + j^2f^2) \\ 2(e^2c^2 + f^2g^2) & -e^4 - 16\Delta_{deg}^2 & 2e^2(g^2 + h^2) & -2(c^2h^2 + f^2j^2) & 2f^2(j^2 + g^2) \\ 2g^2(f^2 + h^2) & 2(a^2d^2 + g^2h^2) & -a^4 - 16\Delta_{eah}^2 & 2a^2(h^2 + i^2) & -2(d^2i^2 + g^2f^2) \\ -2(e^2j^2 + h^2g^2) & 2h^2(g^2 + i^2) & 2(b^2e^2 + h^2i^2) & -b^4 - 16\Delta_{abi}^2 & 2b^2(i^2 + j^2) \\ 2c^2(j^2 + f^2) & -2(a^2f^2 + i^2h^2) & 2i^2(h^2 + j^2) & 2(c^2a^2 + i^2j^2) & -c^4 - 16\Delta_{bcj}^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} -4c^2d^2 & 0 & 0 & 0 & 0 \\ 0 & -4d^2e^2 & 0 & 0 & 0 \\ 0 & 0 & -4e^2a^2 & 0 & 0 \\ 0 & 0 & 0 & -4a^2b^2 & 0 \\ 0 & 0 & 0 & 0 & -4b^2c^2 \end{bmatrix}$$

$$R = \begin{bmatrix} -4j^2g^2 & 0 & 0 & 0 & 0 \\ 0 & -4f^2h^2 & 0 & 0 & 0 \\ 0 & 0 & -4g^2i^2 & 0 & 0 \\ 0 & 0 & 0 & -4h^2j^2 & 0 \\ 0 & 0 & 0 & 0 & -4i^2f^2 \end{bmatrix}$$

$S =$

$$\begin{bmatrix} -a^4 - j^4 - 16\Delta_{ajg}^2 & 2d^2(a^2 + b^2) & 2g^2(a^2 + c^2) & 2(g^2h^2 + d^2a^2) & -2c^2h^2 \\ -2d^2i^2 & -b^4 - f^4 - 16\Delta_{bfh}^2 & 2e^2(b^2 + c^2) & 2h^2(b^2 + d^2) & 2(h^2i^2 + e^2b^2) \\ 2(i^2j^2 + a^2c^2) & -2e^2j^2 & -c^4 - g^4 - 16\Delta_{egi}^2 & 2a^2(c^2 + d^2) & 2i^2(c^2 + e^2) \\ 2j^2(d^2 + a^2) & 2(j^2f^2 + b^2d^2) & -2a^2f^2 & -d^4 - h^4 - 16\Delta_{dhj}^2 & 2b^2(d^2 + e^2) \\ 2c^2(e^2 + a^2) & 2f^2(e^2 + b^2) & 2(f^2g^2 + c^2e^2) & -2b^2g^2 & -e^4 - i^4 - 16\Delta_{eif}^2 \end{bmatrix}$$

Using this one readily discovers that the simplex with edge-lengths given by

$$(a, b, c, d, e, f, g, h, i, j) = (3, 1, 2, 1, 3, 2, 3, 2, 3, 2)$$

has a rational hypervolume of $21/96$. Thus scaling each edge up by a factor of 32 leads to a simplex with an integer hypervolume of 229376.

Since any simplex has 10 edges, 10 faces, 5 volumes and 1 hypervolume the occurrence of a perfect 4–pyramid, which requires all these quantities to be integral, is probably a rarity at best.

6 Regular n -Simplex

Define an n -simplex to be the n dimensional analog of a tetrahedron i.e. $n + 1$ vertices connected by $n(n + 1)/2$ edges such that no set of m vertices (and corresponding edges) can be embedded in an $m - 2$ dimensional subspace. Some associated quantities of an n -simplex are

V_n : hypervolume of a regular n -simplex with edge length a ,

R_n : circumhypersphere radius of a regular n -simplex with edge length a ,

r_n : inhypersphere radius of a regular n -simplex with edge length a ,

h_n : distance from any vertex to circumcentre of the opposite regular $(n - 1)$ -simplex.

The recurrence relations relating these quantities are

$$\begin{aligned} h_n^2 + R_{n-1}^2 &= a^2 \\ V_n &= \frac{1}{n} V_{n-1} h_n \\ V_n &= \frac{n+1}{n} V_{n-1} r_n \\ R_n^2 &= r_n^2 + R_{n-1}^2 \end{aligned}$$

Since $V_1 = h_1 = 1$ and $R_1 = r_1 = 1/2$ one can use induction to prove that the explicit equations for the quantities V_n, R_n, r_n, h_n of an n -dimensional simplex are given by

$$\begin{aligned} V_n &= \frac{a^n}{n!} \sqrt{\frac{n+1}{2^n}}, \\ h_n &= \frac{a}{\sqrt{2}} \sqrt{\frac{n+1}{n}}, \\ R_n &= \frac{a}{\sqrt{2}} \sqrt{\frac{n}{n+1}}, \\ r_n &= \frac{a}{\sqrt{2n(n+1)}}. \end{aligned}$$

Theorem 2 *Any regular, rational-edged, n -simplex has rational hypervolume if and only if n takes the form $4d(d+1)$ or $2d^2 - 1$.*

Proof : If $n = 2m$ then

$$V_{2m} = \frac{a^{2m} \sqrt{2m+1}}{(2m)! 2^m}$$

which is rational (for rational a) whenever $2m+1 = d^2$ say. Since d must be odd let $d = 2D+1$ so that $n = 4D(D+1)$ gives

$$V_{4D(D+1)} = \frac{(2D+1)a^{4D(D+1)}}{(4D(D+1))! 2^{2D(D+1)}}.$$

So for $n = 1, 7, 17, 31, 49, \dots$ dimensional spaces any rational edged n -simplex has rational hypervolume.

If $n = 2m+1$ then

$$V_{2m+1} = \frac{a^{2m+1} \sqrt{m+1}}{(2m+1)! 2^m}$$

which is rational (for rational a) whenever $m+1 = d^2$ say. Thus $n = 2d^2 - 1$ leads to

$$V_{2d^2-1} = \frac{da^{2d^2-1}}{(2d^2-1)! 2^{d^2-1}}.$$

So for $n = 8, 24, 48, 80, \dots$ dimensional spaces any rational edged n -simplex has rational hypervolume. ■

7 References

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