Cyclic pentagons with rational sides and area

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1 Introduction

A cyclic pentagon with rational sides does not necessarily have rational area. For example the regular pentagon with a side length of one is clearly cyclic but has area

$$\frac{1}{4}\sqrt{25+10\sqrt{5}}.$$

If one considers a pentagon with arbitrary side lengths then it is always possible to deform it in such a way that it is convex and all the vertices lie on a circle. This cyclic pentagon has the largest area, among all pentagons with those five edge lengths, and is given by the maximal root of the polynomial

$$\sum_{j=0}^{7} 2^{4j} p_{28-4j}(a,b,c,d,e) A_5^{2j}$$

where the coefficients, p_i, are homogeneous polynomials, of degree i, in the side lengths, a, b, c, d, e, (see [3]). Surprising as it may seem, such pentagons can in fact have rational area, a fact first shown by Euler who provided an infinite family. An example of an integer sided cyclic pentagon with smallest perimeter (for all distinct side lengths) is that with sides $(16, 25, 33, 39, 63)$ and area 1848.

In this paper we are interested in the properties of the diagonals of such cyclic pentagons. Previous theoretical work, [1], revealed that such pentagons can have either zero or five rational diagonals—however all examples found by computational search so far have always turned out to have five rational diagonals. Moreover, it was shown that the diagonals lie in an extension field of $\mathbb{Q}$ of degree no more than seven.

We show that it is possible to restrict the degree of the extension field to no greater than four and then consider some special cases when the degree of the extension is greater than one.

2 The degree four diagonal equation

The area of a cyclic pentagon can be calculated in terms of the sides and a single diagonal by using Heron's formula and Brahmagupta's formula for the area of a triangle and cyclic quadrilateral respectively [2]. If we let A_3 denote the area of a triangle and A_4 that of a cyclic quadrilateral, then recall that:

$$A_3 = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$
$$A_4 = \frac{1}{4}\sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}.$$

By using a diagonal to dissect a rational area cyclic pentagon into a quadrilateral and a triangle one can show that the diagonals are no worse that quartic irrationals.

If we define a Robbins pentagon to be a cyclic pentagon with rational sides and rational area then we obtain the following result.

Theorem 1 *Any diagonal of a Robbins pentagon satisfies a polynomial in* $\mathbb{Z}[x]$ *of degree no greater than four.*

Proof : Let the sides of the pentagon be a, b, c, d, e and the diagonal opposite side a be α. Then α dissects the pentagon, of area A_5, into a triangle, of area A_3, and a quadrilateral, of area A_4, where $A_5 = A_3 + A_4$. Rather than view this as an area formula we assume that $a, b, c, d, e, A_5 \in \mathbb{Q}$ and use the

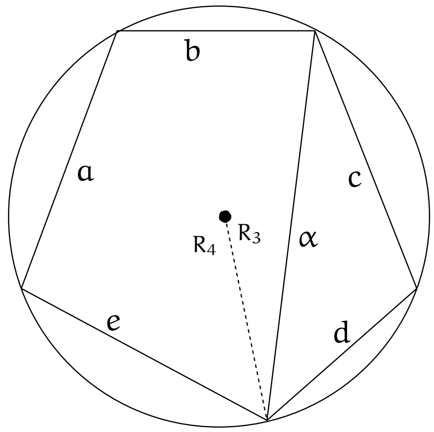

Figure 1: One diagonal in a pentagon

equations for A_3, A_4 and $A_5 = A_3 + A_4$ to express the lone diagonal α as a polynomial in the other six variables.

Thus we need to consider the system

$$\begin{aligned}
A_3 &= \frac{1}{4}\sqrt{(c+d+\alpha)(-c+d+\alpha)(c-d+\alpha)(c+d-\alpha)} \\
A_4 &= \frac{1}{4}\sqrt{(-a+b+e+\alpha)(a-b+e+\alpha)(a+b-e+\alpha)(a+b+e-\alpha)} \\
A_5 &= A_3 + A_4.
\end{aligned}$$

In the last equation, we square A_5 and collect rational parts to one side and then square again to obtain a polynomial that the diagonal satisfies, namely,

$$(A_5^2 - (A_3^2 + A_4^2))^2 - 4A_3^2A_4^2.$$

In particular this means that α is a root of the degree four polynomial

$$f_4(\alpha) = c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4$$

where the coefficients, given by

$$\begin{aligned}
c_0 &= 2^8A_5^4 + 2^5(a^4+b^4+c^4+d^4+e^4-2a^2b^2-2a^2e^2-2b^2e^2-2c^2d^2)A_5^2 \\
&\quad + (a^4+b^4-c^4-d^4+e^4-2a^2b^2-2a^2e^2-2b^2e^2+2c^2d^2)^2 \\
c_1 &= 2^4abe[2^4A_5^2+a^4+b^4-c^4-d^4+e^4-2(a^2b^2+a^2e^2+b^2e^2-c^2d^2)] \\
c_2 &= -2^6(a^2+b^2+c^2+d^2+e^2)A_5^2+2^6a^2b^2e^2-4(a^2+b^2-c^2-d^2+e^2) \\
&\quad \times (a^4+b^4-c^4-d^4+e^4-2a^2b^2-2a^2e^2-2b^2e^2+2c^2d^2) \\
c_3 &= 2^5abe(a^2+b^2-c^2-d^2+e^2) \\
c_4 &= 2^6A_5^2+2^2(a^2+b^2-c^2-d^2+e^2)^2
\end{aligned}$$

are polynomials in the sides and the area, thus are clearly all rational. □

At this point we realised that any diagonal satisfies two polynomials, namely $f_4(\alpha)$ described above and the degree seven polynomial (found by MacDougall and Buchholz [1]), denoted by $f_7(\alpha)$. Moreover, we can use a resultant computation to eliminate α from these two equations. Since

$$\mathcal{RES}(f_4(\alpha), f_7(\alpha), \alpha) \in \mathbb{Q}[a, b, c, d, e, A_5]$$

we can also use Robbins' formula for the area of a cyclic pentagon to produce a single equation constraining a, b, c, d, e—this is a plausible path for proving that any diagonals of any Robbins pentagon is rational.

3 General results

In this section we determine the effect of the field, in which a diagonal lies, on the areas of the three non-overlapping triangular regions in a cyclic pentagon. First we require an intermediate result that holds independently of any reference to the pentagon.

Lemma 1 *Let* K *be a real number field. If* $A^2, B^2, C^2, A+B+C \in K$ *and* $(B+C)(A+C)(A-B)(A+B)(A+B-3C)(A+B+C) \neq 0$ *then* $A, B, C \in K$.

Proof: If we let $A + B + C = r$ where $r \in K$ then we simply rewrite the 3 equations obtained from r, r^3, r^5 as 3 linear equations in A, B, C. Notice that

$$\begin{aligned} r &= A + B + C \\ r^3 &= a_3 A + b_3 B + c_3 C + 6ABC \\ r^5 &= a_5 A + b_5 B + c_5 C + d_5 ABC \end{aligned} \tag{1}$$

where

$$\begin{aligned} a_3 &= A^2 + 3B^2 + 3C^2 \\ b_3 &= 3A^2 + B^2 + 3C^2 \\ c_3 &= 3A^2 + 3B^2 + C^2 \\ a_5 &= A^4 + 5B^4 + 5C^4 + 10A^2B^2 + 10A^2C^2 + 30B^2C^2 \\ b_5 &= 5A^4 + B^4 + 5C^4 + 10A^2B^2 + 30A^2C^2 + 10B^2C^2 \\ c_5 &= 5A^4 + 5B^4 + C^4 + 30A^2B^2 + 10A^2C^2 + 10B^2C^2 \\ d_5 &= 20A^2 + 20B^2 + 20C^2 \end{aligned}$$

and by hypothesis $a_3, b_3, c_3, a_5, b_5, c_5, d_5 \in K$. Now use the identity

$$ABC = -C^2A - C^2B + \left[\frac{(A+B+C)^2 - (A^2+B^2+C^2)}{2}\right] C$$

to rewrite the product ABC in terms of A, B, C and substitute into equations (1) we get

$$\begin{bmatrix} r \\ r^3 \\ r^5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \overline{a}_3 & \overline{b}_3 & \overline{c}_3 \\ \overline{a}_5 & \overline{b}_5 & \overline{c}_5 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

where the entries in the matrix, denoted by M say, are all in K. The determinant of M is given by

$$\det(M) = (B + C)(A + C)(A - B)(A + B)(A + B - 3C)(A + B + C)$$

which must be non-zero for the system to have a solution. $\square$

Using this lemma we can restrict the areas of the triangles in a Robbins pentagon.

Theorem 2 *Let* K *be a real number field. If any diagonal in a Robbins pentagon lies in* K *then the areas of any three non-overlapping triangular regions bounded by sides and any 2 non-intersecting diagonals also lie in* K*.*

Proof : First note that, by Theorem 11 of [1], if one diagonal lies in K then they all lie in K. If we denote the three triangular areas by A, B, C then by Heron's formula it is clear that $A^2, B^2, C^2 \in K$. Since $A + B + C \in \mathbb{Q} \subseteq K$, then by Lemma 1 we only need to show that

$$(B + C)(A + C)(A - B)(A + B)(A + B - 3C)(A + B + C) \neq 0.$$

Clearly, since A, B, C are positive real values we need only check the cases $A = B$ and $A+B = 3C$. The former case reduces to showing that $A^2, C^2, 2A+C \in K$ implies that $A, C \in K$. We let $2A + C = r$ and then observe that r and r^3 lead to the equation

$$\begin{bmatrix} r \\ r^3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 8A^2 + 6C^2 & 12A^2 + C^2 \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}$$

which has a unique solution precisely when the determinant $4(2A - C)(2A + C)$ is non-zero. If $C = 2A$ then $2A + C = 4A \in K$ implies that $A \in K$. The other case, namely $C = 2A$ is not possible for positive areas.

In the case when $A + B = 3C$ we have $C = \frac{1}{4}(4C) = \frac{1}{4}(A + B + C) \in K$ so we only need to show that $A^2, B^2, A + B \in K$ implies $A, B \in K$. This is identical to the previous (2×2) proof. □

References

[1] Ralph H. Buchholz and James A. MacDougall, *Cyclic Polygons with Rational Sides and Area*, Journal of Number Theory, vol. 128, pp. 17-48 (2008).

[2] Leonard E. Dickson, *History of the Theory of Numbers*, volume 2, Chelsea, (1952).

[3] D. P. Robbins, *Areas of polygons inscribed in a circle*, American Mathematical Monthly, June-July, (1995).