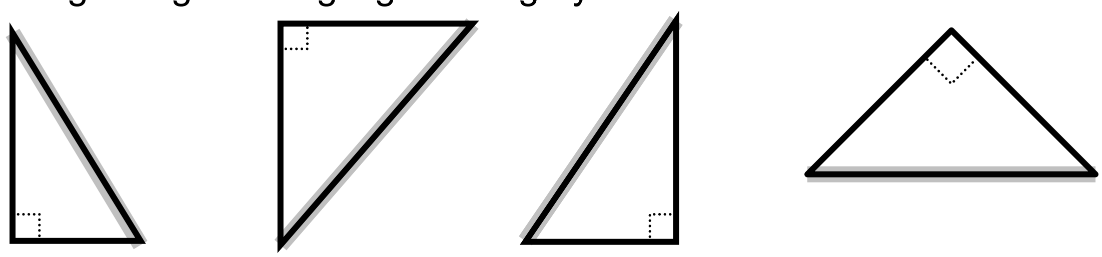
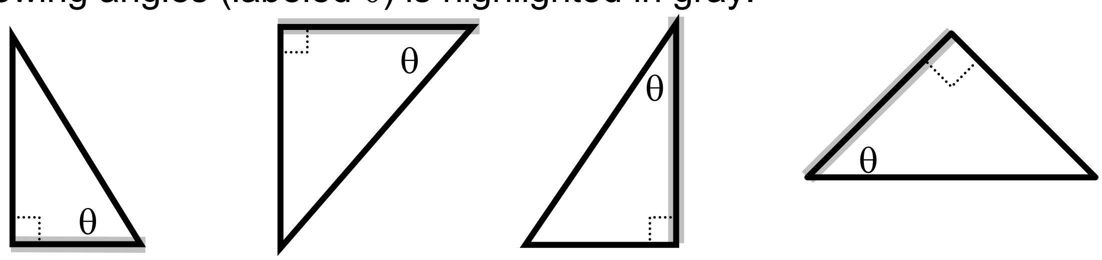
Sect 10.2 – Trigonometric Ratios

Objective 1: Understanding Adjacent, Hypotenuse, and Opposite sides of an acute angle in a right triangle.

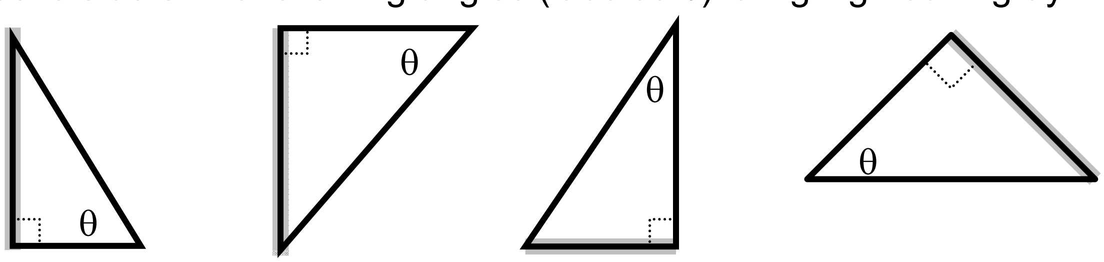
In a right triangle, the hypotenuse is always the longest side; it is the side opposite of (or "across from") the right angle. The hypotenuse of the following triangles is highlighted in gray.



The adjacent side of an acute angle is the side of the triangle that forms the acute angle with the hypotenuse. The adjacent side is the side of the angle that is "adjacent" to or next to the hypotenuse. The adjacent side depends on which of the acute angles one is using. The adjacent side of the following angles (labeled θ) is highlighted in gray.

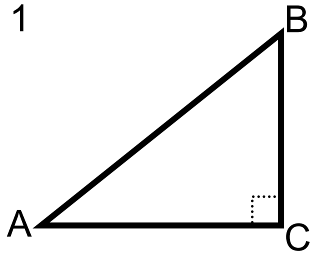


The opposite side of an acute angle is the side of the triangle that is <u>not</u> a side of the acute angle. It is "across from" or opposite of the acute angle. The opposite side depends on which of the acute angles one is using. The opposite side of the following angles (labeled θ) is highlighted in gray.



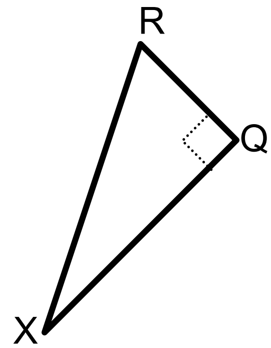
<u>For each acute angle in the following right triangles, identify the adjacent, hypotenuse, and opposite side:</u>

Ex. 1



<u>Solution:</u>
For $\angle$ A,
the adjacent side is $\overline{AC}$,
the opposite side is $\overline{BC}$,
and the hypotenuse is $\overline{AB}$.
For $\angle$ B,
the adjacent side is $\overline{BC}$,
the opposite side is $\overline{AC}$,
and the hypotenuse is $\overline{AB}$.

Ex. 2

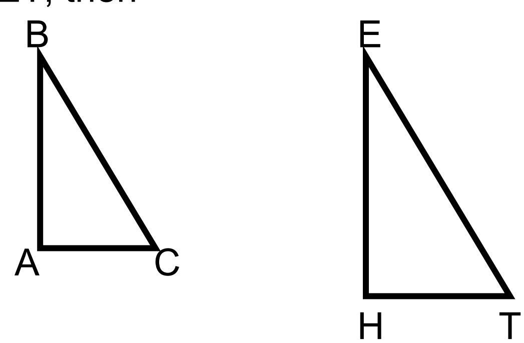


<u>Solution:</u>
For $\angle$ R,
the adjacent side is $\overline{QR}$,
the opposite side is $\overline{QX}$,
and the hypotenuse is $\overline{RX}$.
For $\angle$ X
the adjacent side is $\overline{QX}$,
the opposite side is $\overline{QR}$,
and the hypotenuse is $\overline{RX}$.

Objective 2: Understanding Trigonometric Ratios or Functions.

Recall that two triangles are similar if they have the same shape, but are not necessarily the same size. In similar triangles, the corresponding angles are equal. Thus, if $\Delta ABC \sim \Delta HET$, then $m\angle A = m\angle H$, $m\angle B = m\angle E$, and $m\angle C = m\angle T$. The corresponding sides are not equal. However, the ratios of corresponding sides are equal since one triangle is in proportion to the other triangle. Thus, if $\Delta ABC \sim \Delta HET$, then

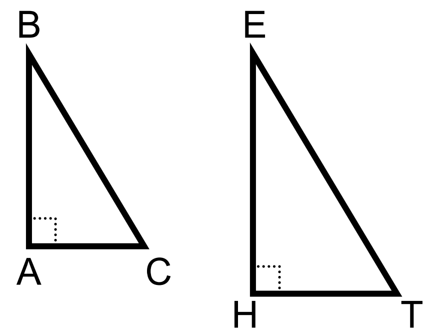
$$\frac{AB}{HE} = \frac{BC}{ET} = \frac{AC}{HT}$$



Now, let's consider the proportion $\frac{AB}{HE} = \frac{BC}{ET}$. If we cross multiply, we get $(AB)\bullet(ET) = (HE)\bullet(BC)$. Now, divide both sides by $(BC)\bullet(ET)$ and simplifying, we get:

$$\frac{(AB)\bullet(ET)}{(BC)\bullet(ET)} = \frac{(HE)\bullet(BC)}{(BC)\bullet(ET)} \Longrightarrow \frac{AB}{BC} = \frac{HE}{ET}$$

Now consider angles $\angle C$ and $\angle T$.
If $\angle A$ and $\angle H$ are right angles, then the proportion $\frac{AB}{BC} = \frac{HE}{ET}$ implies that the ratio of the opposite side to the hypotenuse is the same for any right triangle if the corresponding acute angles have the same measure.

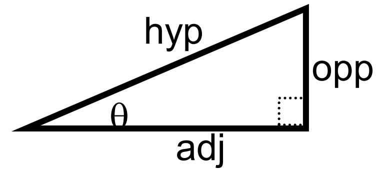


We will define this ratio as the **Sine Ratio** or **Sine Function**. The abbreviation for the sine function is "sin", but it is pronounced as "sign".

<u>Sine Ratio or Function</u>

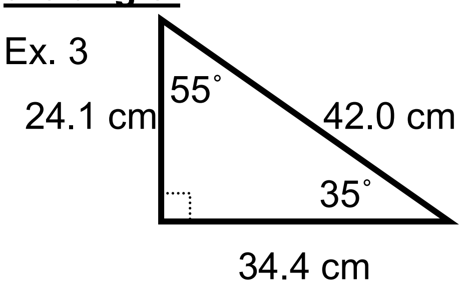
Given an acute angle θ in a right triangle, then the sine ratio of θ, is the ratio of the opposite side of θ to the hypotenuse.

$$\sin\theta = \frac{\text{oppositeside}}{\text{hypotenuse}} = \frac{\text{opp}}{\text{hyp}}$$



<u>For each acute angle in the following right triangles, find the sine of the angle:</u>

Ex. 3



<u>Solution:</u>

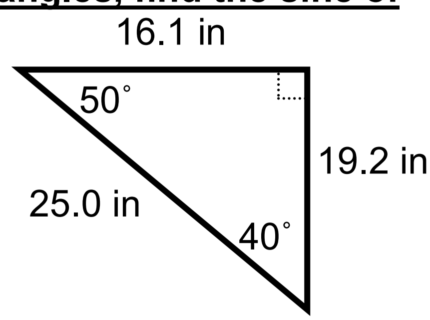
$\sin 35^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{24.1}{42.0}$

$= 0.5738\ldots \approx 0.574$

$\sin 55^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{34.4}{42.0}$

$= 0.8190\ldots \approx 0.819$

Ex. 4



<u>Solution:</u>

$\sin 40^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{16.1}{25.0}$

$= 0.644$

$\sin 50^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{19.2}{25.0}$

$= 0.768$

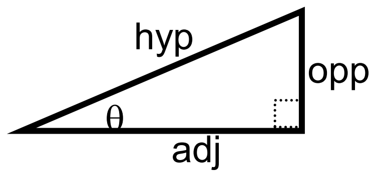
Since the opposite is never greater than the hypotenuse, then the value of the sine function cannot exceed 1. If the opposite side is equal to the hypotenuse, then the measure of the angle is 90˚ and the triangle degenerates into a vertical line. This means that $\sin 90^\circ = 1$. Similarly, if the opposite side has zero length meaning the sine ratio is 0, then the measure of the angle is 0˚ and the triangle degenerates into a horizontal line. Hence, $\sin 0^\circ = 0$.

We have seen that if the corresponding acute angles of rights triangles have the same measure, the ratio of the opposite side to the hypotenuse is fixed. We can make the same argument that the ratio of the adjacent side to the hypotenuse is also fixed. We will define this ratio as the **Cosine Ratio** or **Cosine Function**. The abbreviation for the cosine function is "cos", but it is pronounced as "co-sign".

<u>Cosine Ratio or Function</u>

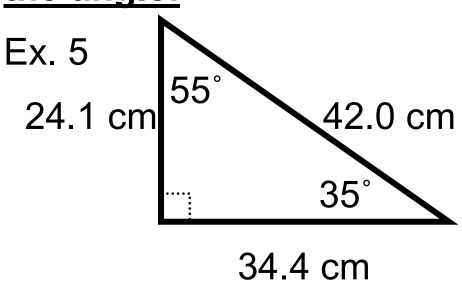
Given an acute angle θ in a right triangle, then the cosine ratio of θ, is the ratio of the adjacent side of θ to the hypotenuse.

$$\cos\theta = \frac{\text{adjacentside}}{\text{hypotenuse}} = \frac{\text{adj}}{\text{hyp}}$$



<u>For each acute angle in the following right triangles, find the cosine of the angle:</u>

Ex. 5



<u>Solution:</u>

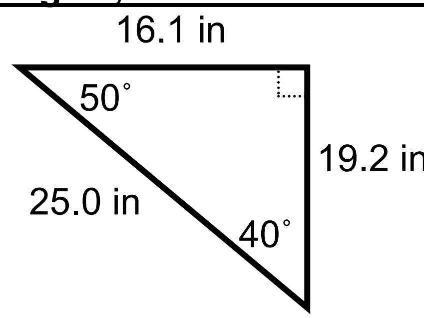
$\cos 35^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{34.4}{42.0}$

$= 0.8190\ldots \approx 0.819$

$\cos 55^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{24.1}{42.0}$

$= 0.5738\ldots \approx 0.574$

Ex. 6



<u>Solution:</u>

$\cos 40^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{19.2}{25.0}$

$= 0.768$

$\cos 50^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{16.1}{25.0}$

$= 0.644$

Since the adjacent is never greater than the hypotenuse, then the value of the cosine function cannot exceed 1. If the adjacent side is equal to the hypotenuse, then the measure of the angle is 0˚ and the triangle degenerates into a horizontal line. This means that cos 0˚ = 1. Similarly, if the adjacent side has zero length meaning the cosine ratio is 0, then the measure of the angle is 90˚ and the triangle degenerates into a vertical line. Hence, cos 90˚ = 0.

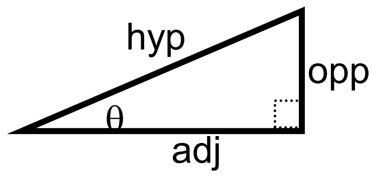
We have seen that if the corresponding acute angles of rights triangles have the same measure, the ratio of the opposite side to the hypotenuse is

fixed and the ratio of the adjacent side to the hypotenuse is fixed. We can make the same argument that the ratio of the opposite side to the adjacent side is also fixed. We will define this ratio as the **Tangent Ratio** or **Tangent Function**. The abbreviation for the tangent function is "tan", but it is pronounced as "tangent".

Tangent Ratio or Function

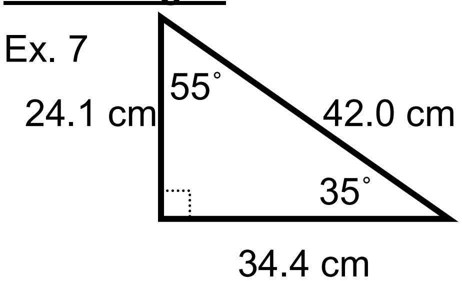
Given an acute angle θ in a right triangle, then the tangent ratio of θ, is the ratio of the opposite side of θ to the adjacent side of θ.

$$\tan\theta = \frac{\text{oppositeside}}{\text{adjacentside}} = \frac{\text{opp}}{\text{adj}}$$



For each acute angle in the following right triangles, find the tangent of the angle:

Ex. 7



Solution:

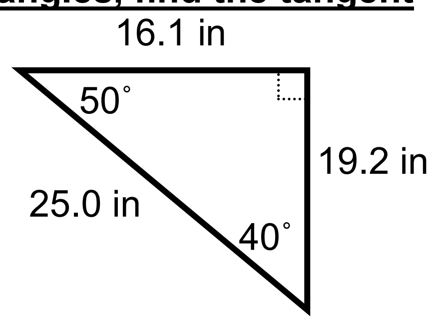
$\tan 35^{\circ} = \frac{\text{opp}}{\text{adj}} = \frac{24.1}{34.4}$

$= 0.7005\ldots \approx 0.701$

$\tan 55^{\circ} = \frac{\text{opp}}{\text{adj}} = \frac{34.4}{24.1}$

$= 1.427\ldots \approx 1.43$

Ex. 8



Solution:

$\tan 40^{\circ} = \frac{\text{opp}}{\text{adj}} = \frac{16.1}{19.2}$

$\approx 0.8385\ldots = 0.839$

$\tan 50^{\circ} = \frac{\text{opp}}{\text{adj}} = \frac{19.2}{16.1}$

$= 1.192\ldots \approx 1.19$

When the opposite side has zero length, then the measure of the angle is zero. The tangent ratio is the ratio of 0 to the adjacent side, which is equal to zero. Hence, $\tan 0^{\circ} = 0$. Similarly, if the adjacent side has zero length, then the measure of the angle is 90˚. The tangent ratio is the ratio of the opposite side to 0, which is undefined. Thus, $\tan 90^{\circ}$ is undefined. Unlike the sine and cosine function, the values of the tangent function can exceed one.

There are some memories aids that you can use to remember the trigonometric ratios. If you remember the order sine, cosine, and tangent, then you might find one of these phrases helpful:

Trigonometric Ratio	Ratio of sides	Rated G Phrase	Rated PG-13 Phrase
sin θ =	$\frac{\text{opp}}{\text{hyp}}$	$\frac{\text{oscar}}{\text{had}}$	$\frac{\text{oh}}{\text{h*ll}}$
cos θ =	$\frac{\text{adj}}{\text{hyp}}$	$\frac{\text{a}}{\text{heap}}$	$\frac{\text{another}}{\text{hour}}$
Tan θ =	$\frac{\text{opp}}{\text{adj}}$	$\frac{\text{of}}{\text{apples}}$	$\frac{\text{of}}{\text{algebra}}$

If you want to include the trigonometric functions themselves, you can use the following phrases:

s-o-h-c-a-h-t-o-a

s	**o**	**h**	**c**	**a**	**h**	**t**	**o**	**a**
i	p	y	o	d	y	a	p	d
n	p	p	s	j	p	n	p	j
e	o	o	i	a	o	g	o	a
	s	t	n	c	t	e	s	c
	i	e	e	e	e	n	i	e
	t	n		n	n	t	t	n
	e	u		t	u		e	t
		s			s			
		e			e			

sinful oscar had consumed a heaping taste of apples

sinful	**o**scar	**h**ad	**c**onsumed	**a**	**h**eaping	**t**aste	**o**f	**a**pples.
i	p	y	o	d	y	a	p	d
n	p	p	s	j	p	n	p	j
e	o	o	i	a	o	g	o	a
	s	t	n	c	t	e	s	c
	i	e	e	e	e	n	i	e
	t	n		n	n	t	t	n
	e	u		t	u		e	t
		s			s			
		e			e			

Complete the following table:

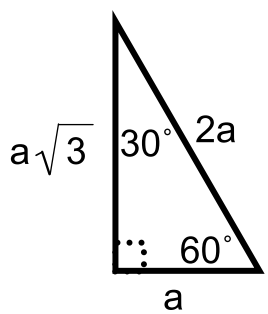
Ex. 9

θ	0°	30°	36.9°	45°	53.1°	60°	90°
$\sin\theta$							
$\cos\theta$							
$\tan\theta$							

Solution:
Recall that $\sin 0^\circ = 0$, $\cos 0^\circ = 1$, and $\tan 0^\circ = 0$.
Also, $\sin 90^\circ = 1$, $\cos 90^\circ = 0$, and $\tan 90^\circ$ is undefined:

θ	0°	30°	36.9°	45°	53.1°	60°	90°
$\sin\theta$	0						1
$\cos\theta$	1						0
$\tan\theta$	0						undef.

One of our special triangles from before was a 30-60-90 triangle:



$$\sin 30^\circ = \frac{opp}{hyp} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{adj}{hyp} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{opp}{adj} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

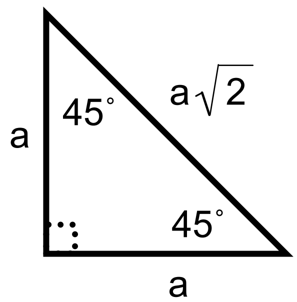
$$\sin 60^\circ = \frac{opp}{hyp} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{adj}{hyp} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{opp}{adj} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

θ	0°	30°	36.9°	45°	53.1°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$				$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$				$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$				$\sqrt{3}$	undef.

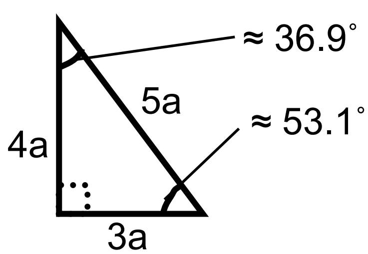
Another of our special triangles was a 45-45-90 triangle:



$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{a}{a} = 1$$

θ	0°	30°	36.9°	45°	53.1°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$		$\frac{1}{\sqrt{2}}$		$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$		$\frac{1}{\sqrt{2}}$		$\frac{1}{2}$	0
Tan θ	0	$\frac{1}{\sqrt{3}}$		1		$\sqrt{3}$	undef.

Our last special triangle was a 3-4-5 triangle:



$$\sin 36.9^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{3a}{5a} = \frac{3}{5}$$
$$\cos 36.9^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{4a}{5a} = \frac{4}{5}$$
$$\tan 36.9^\circ = \frac{\text{opp}}{\text{adj}} = \frac{3a}{4a} = \frac{3}{4}$$
$$\sin 53.1^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{4a}{5a} = \frac{4}{5}$$
$$\cos 53.1^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{3a}{5a} = \frac{3}{5}$$
$$\tan 53.1^\circ = \frac{\text{opp}}{\text{adj}} = \frac{4a}{3a} = \frac{4}{3}$$

θ	0°	30°	36.9°	45°	53.1°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2} \approx 0.866$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}} \approx 0.707$	$\frac{3}{5}$	$\frac{1}{2}$	0
Tan θ	0	$\frac{1}{\sqrt{3}} \approx 0.557$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3} \approx 1.73$	undef.

Objective 3: Evaluating trigonometric ratios on a scientific calculator.

On a scientific calculator, SIN key is for the sine ratio, COS key is for the cosine ratio, and TAN key is for the tangent ratio. To evaluate a trigonometric ratio for a specific angle, you first want to be in the correct mode. If the angle is in degree, then the calculator needs to be in degree mode while if the angle is in radians, the calculator needs to be in radian mode. Next, type in the angle and then hit the appropriate trigonometric ratio. On some calculators, you might need to enter the trigonometric ratio first and then the angle.

<u>Evaluate the following (round to three significant digits):</u>

Ex. 10a $\cos 26^\circ$

Ex. 10b $\tan 78.3^\circ$

Ex. 10c $\sin 16\frac{2}{3}^\circ$

Ex. 10d $\sin 56^\circ 11'$

Ex. 10e $\tan 89^\circ 9'$

Ex. 10f $\cos 147^\circ 17'$

Ex. 10g $\sin 1.2$

Ex. 10h $\tan \frac{3\pi}{2}$

<u>Solution:</u>

a) We put our calculator in degree mode, type 26 and hit the COS key: $\cos 26^\circ = 0.8987\ldots \approx 0.899$

b) We put our calculator in degree mode, type 78.3 and hit the TAN key: $\tan 78.3^\circ = 4.828\ldots \approx 4.83$

c) We put our calculator in degree mode, type $16\frac{2}{3}$ and hit the SIN key: $\sin 16\frac{2}{3}^\circ = 0.2868\ldots \approx 0.287$

d) We put our calculator in degree mode, but we will need to convert the degrees and minutes. We type 56.11 and hit DMS→DD key to get 56.18333…
Now, hit SIN key: $\sin 56^\circ 11' = 0.8308\ldots \approx 0.831$

e) We put our calculator in degree mode, but we will need to convert the degrees and minutes. We type 89.09 and hit DMS→DD key to get 89.15.
Now, hit TAN key: $\tan 89^\circ 9' = 67.40\ldots \approx 67.4$

f) We put our calculator in degree mode, but we will need to convert the degrees and minutes. We type 147.17 and hit DMS→DD key to get 147.28333…
Now, hit COS key: $\cos 147^\circ 17' = -0.8413\ldots \approx -0.841$

g) Since there is no degree mark on the angle, then the angle is in radians. We put our calculator in radian mode, type 1.2 and hit the SIN key: $\sin 1.2 = 0.9320\ldots \approx 0.932$

h) Since there is no degree mark on the angle, then the angle is in radians. We put our calculator in radian mode, type $\frac{3\pi}{2}$ and hit the TAN key: $\tan \frac{3\pi}{2}$ = Error.

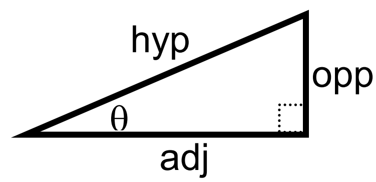
This means that $\tan \frac{3\pi}{2}$ is undefined.

Objective 4: Finding Angles using Trigonometric Ratio.

If we know the ratio of two sides of a right triangle, we can use inverse trigonometric ratios or functions to find the angle. The inverse trigonometric function produces an angle for the answer.

Inverse Trigonometric Functions

Given a right triangle, we can find the acute angle θ using any of the following:



1) Inverse sine (arcsine)

$$\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right)$$

2) Inverse cosine (arccosine)

$$\theta = \cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right)$$

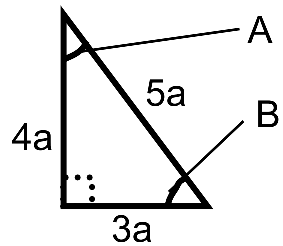
3) Inverse tangent (arctangent)

$$\theta = \tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right)$$

On a scientific calculator, these functions are labeled SIN^{-1}, COS^{-1} and TAN^{-1}. They are usually directly above the SIN, COS, and TAN keys. To find the angle on the calculator, first determine if you want the answer in degrees or in radians and set your calculator in the appropriate mode. Second, type the ratio of the two sides you are given and hit the 2nd (or INV) and then the appropriate trigonometric ratio key. On some scientific calculators, you might have to enter the ratio after hitting the 2nd (or INV) and the appropriate trigonometric ratio key. It is interesting to note that the inverse sine and the inverse tangent on a scientific calculator will produce an answer between -90° and 90° inclusively, whereas the inverse cosine will produce and answer between 0° and 180° inclusively. For acute angles of right triangles, the calculator will yield the desire result, but outside of that context, one needs to be careful.

<u>Verify that the acute angles of a 3-4-5 triangle are ≈ 36.9˚ and 53.1˚:</u>

Ex. 11



<u>Solution:</u>

Since we have all three sides, we can use any of the inverse trigonometric functions. Let's start with ∠A and we will use the inverse sine function.

$m\angle A = \sin^{-1}(\frac{opp}{hyp}) = \sin^{-1}(\frac{3a}{5a}) = \sin^{-1}(\frac{3}{5})$

Enter 3/5 and hit 2nd SIN:

$\sin^{-1}(\frac{3}{5}) = 36.86\ldots \approx 36.9^\circ$.

For ∠B, let's use the inverse tangent function:

$m\angle B = \tan^{-1}(\frac{opp}{adj}) = \tan^{-1}(\frac{4a}{3a}) = \tan^{-1}(\frac{4}{3})$

Enter 4/3 and hit 2nd TAN:

$\tan^{-1}(\frac{4}{3}) = 53.13\ldots \approx 53.1^\circ$.

If we had use the inverse cosine for ∠B, we would have derived the same result:

$m\angle B = \cos^{-1}(\frac{adj}{hyp}) = \cos^{-1}(\frac{3a}{5a}) = \cos^{-1}(\frac{3}{5})$

Enter 3/5 and hit 2nd COS:

$\cos^{-1}(\frac{3}{5}) = 53.13\ldots \approx 53.1^\circ$.

<u>Solve the following. Round to the nearest minute:</u>

Ex. 12a sin x = 0.833 Ex. 12b cos x = 0.528

Ex. 12c tan x = 1.82

Solution:

a) Since $\sin x = 0.833$, then $x = \sin^{-1}(0.833)$.
Type 0.833 and hit 2nd SIN: $\sin^{-1}(0.833) = 56.40\ldots^\circ$
To convert to DMS, hit DD→DMS (2nd =)
$56.40\ldots^\circ = 56^\circ 24' 29'' 3$, but 29" is < 30" so we round down.
$\approx 56^\circ 24'$

b) Since $\cos x = 0.528$, then $x = \cos^{-1}(0.528)$.
Type 0.528 and hit 2nd COS: $\cos^{-1}(0.528) = 58.12\ldots^\circ$
To convert to DMS, hit DD→DMS (2nd =)
$58.12\ldots^\circ = 58^\circ 07' 46'' 4$, but 46" ≥ 30", so we round up
$\approx 58^\circ 8'$

c) Since $\tan x = 1.82$, then $x = \tan^{-1}(1.82)$.
Type 1.82 and hit 2nd TAN: $\tan^{-1}(1.82) = 61.21\ldots^\circ$
To convert to DMS, hit DD→DMS (2nd =)
$61.21\ldots^\circ = 61^\circ 12' 48'' 1$, but 48" is ≥ 30" so we round up.
$\approx 61^\circ 13'$