ENGINEERING CALCULUS II
MTH 1212

Parametric Equation

by

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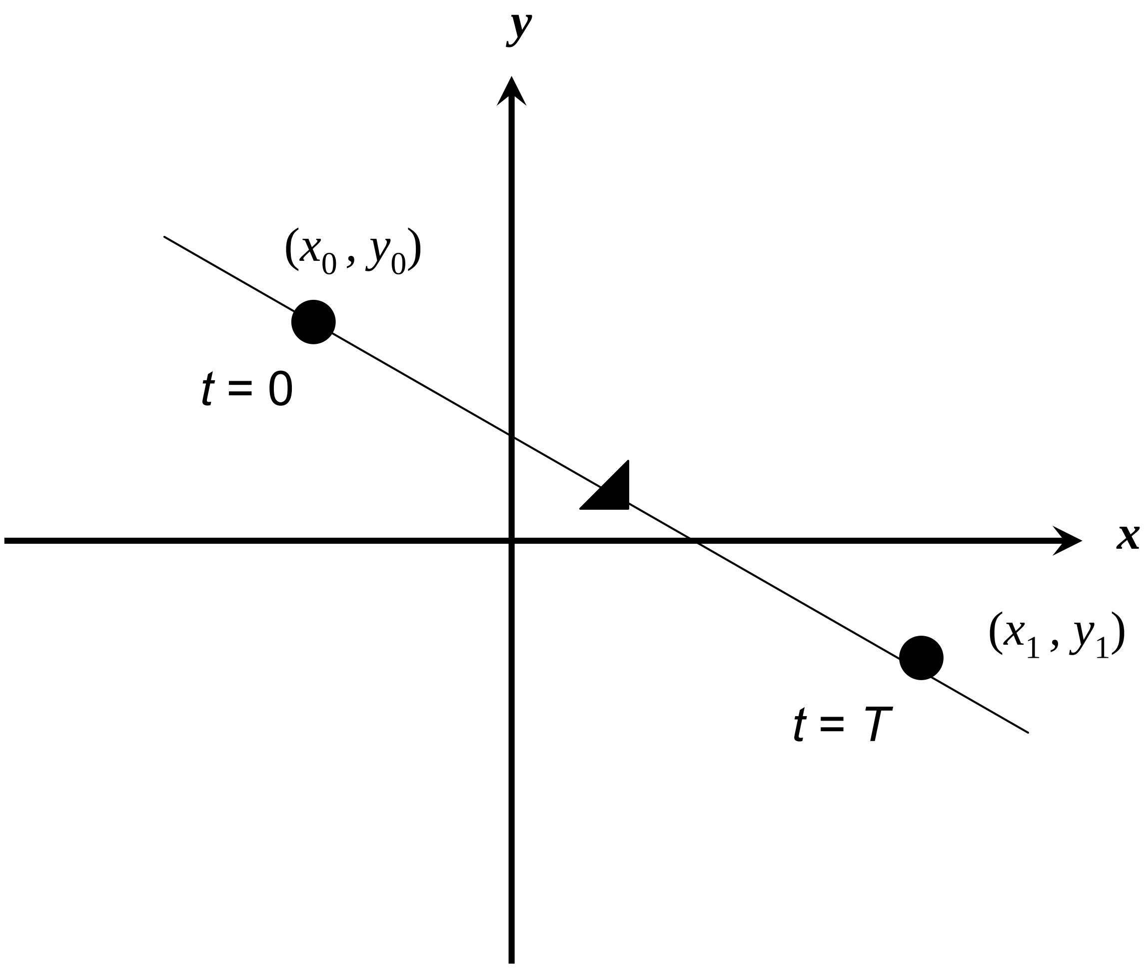
Parametric equation

Usually, we are dealing with x and y parameters. Eg:

- $y = mx + c$ *for linear equation*
- $y = ax^2 + bx + c$ *for parabolic equation*
- $x^2/a^2 + y^2/b^2 = 1$ *for circles & eclipses*

In parametric equations, parameter t is introduced for both x and y parameters to become x(t) and y(t) since the motion of particle need to be investigated.

Parametric equation (line segment)



Parametric equation (line segment)

$$x = a + bt \qquad y = c + dt$$

Initial point (x_0, y_0) *and end point* (x_1, y_1) *are given*

At initial point, $t = 0$

$$x_0 = a + b(0) \qquad y_0 = c + d(0)$$

Hence, $a = x_0 \quad c = y_0$

At end point, $t = T$

$$x_1 = a + bT \qquad y_1 = c + dT$$

$$x_1 = x_0 + bT \qquad y_1 = y_0 + dT$$

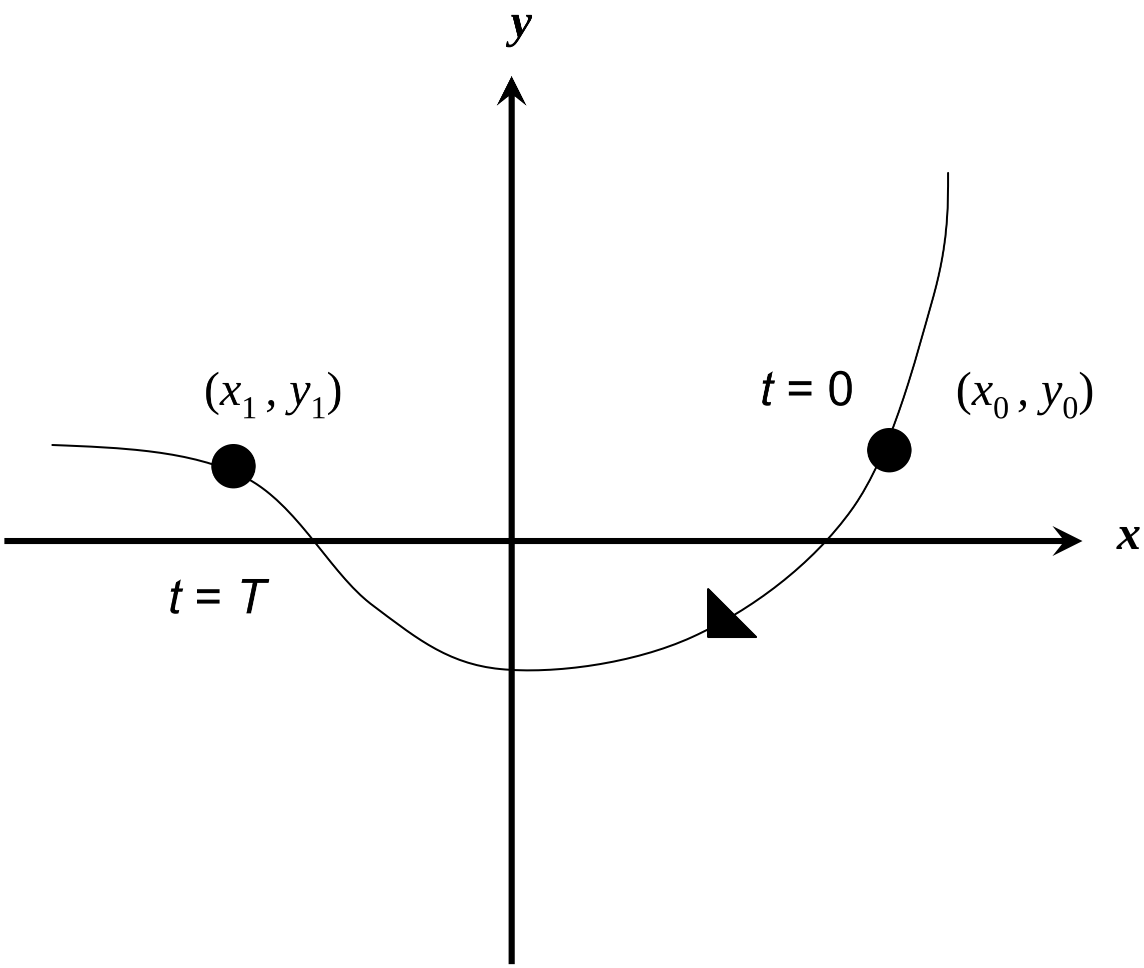
Hence $b = (x_1 – x_0) / T \qquad d = (y_1 – y_0) / T$

Substitute a, b, c, d into x(t) and y(t)

$$x = x_0 + [(x_1 – x_0) / T\ ]\ t \qquad y = y_0 + [(y_1 – y_0) / T]\ t$$

for $0 < t < T$

Parametric equation (curve)



Parametric equation (curve)

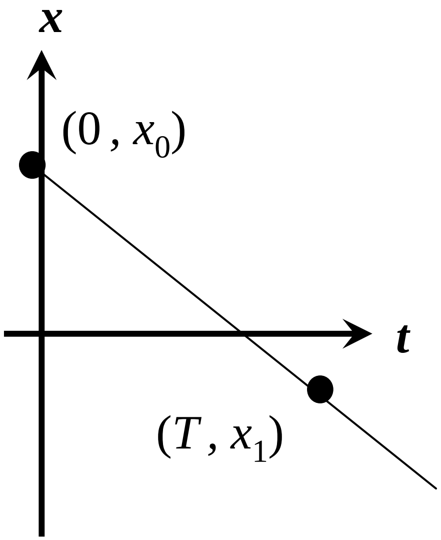
y(x) is given

Initial point (x_0,y_0) and end point (x_1,y_1) are given

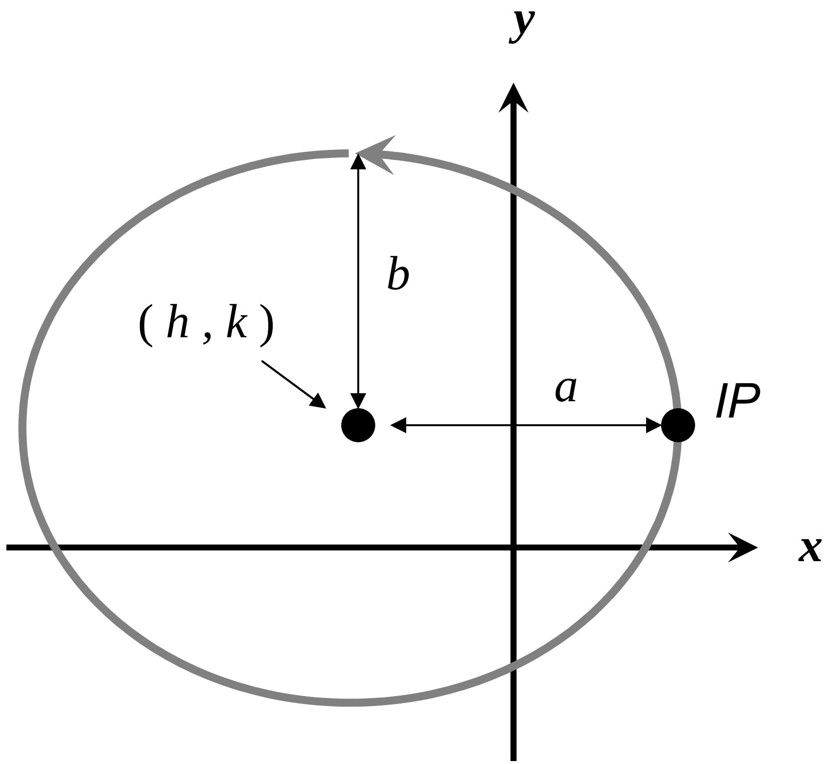
Find the relation between x and t to form x(t)

Substitute x(t) into y(x) to form y(t)

t	x
0	x_0
1	:
:	:
:	:
T	x_1



Parametric equation (circles & eclipses)



$$x = h + a\cos vt$$
$$y = k + b\sin vt$$
$$0 \le vt \le 2\pi$$

(h,k) = center of circle @ eclipse

v = particle speed constant. increasing v causes increasing particle speed

a = horizontal radius

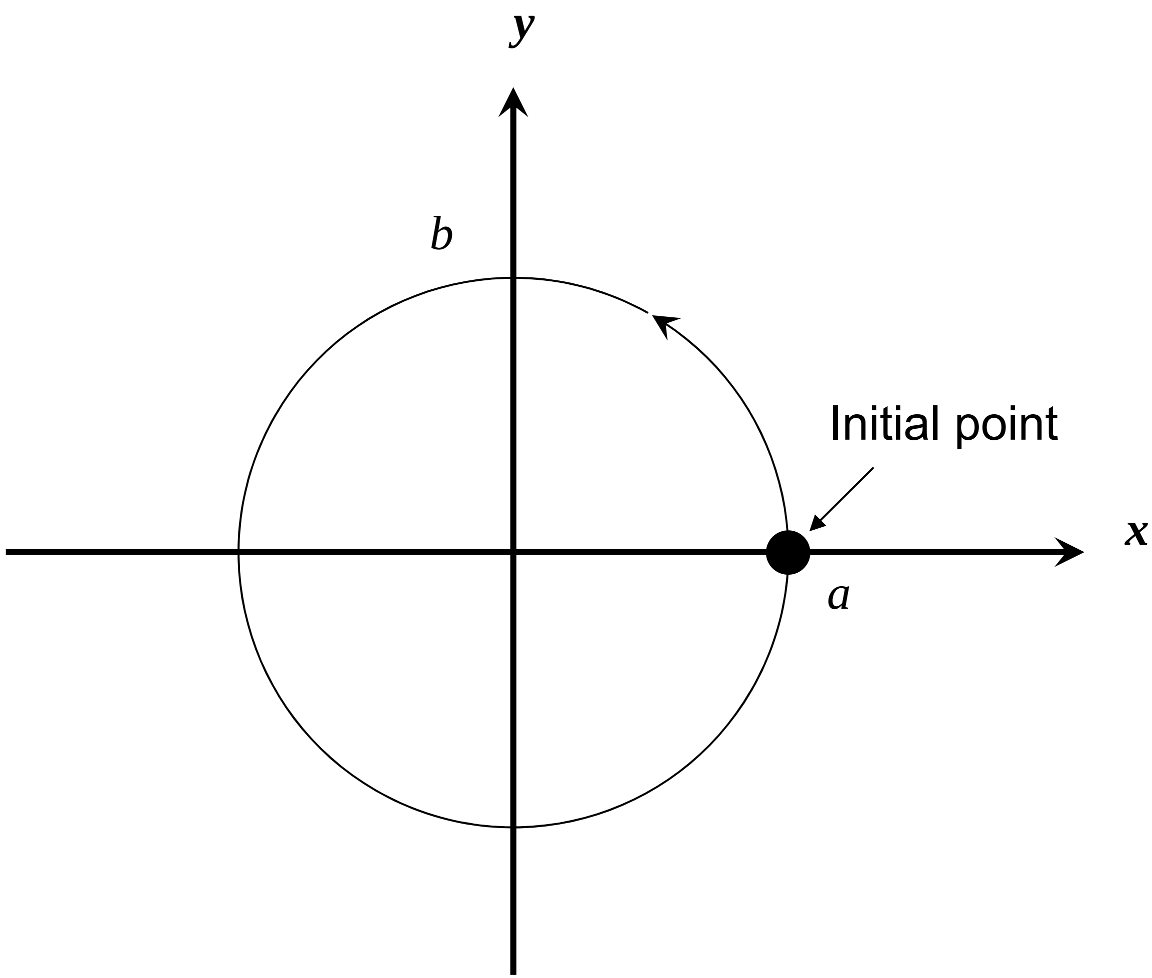
b = vertical radius

Circle → a = b

Eclipse → a ≠ b

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Parametric equation (circles & eclipses)

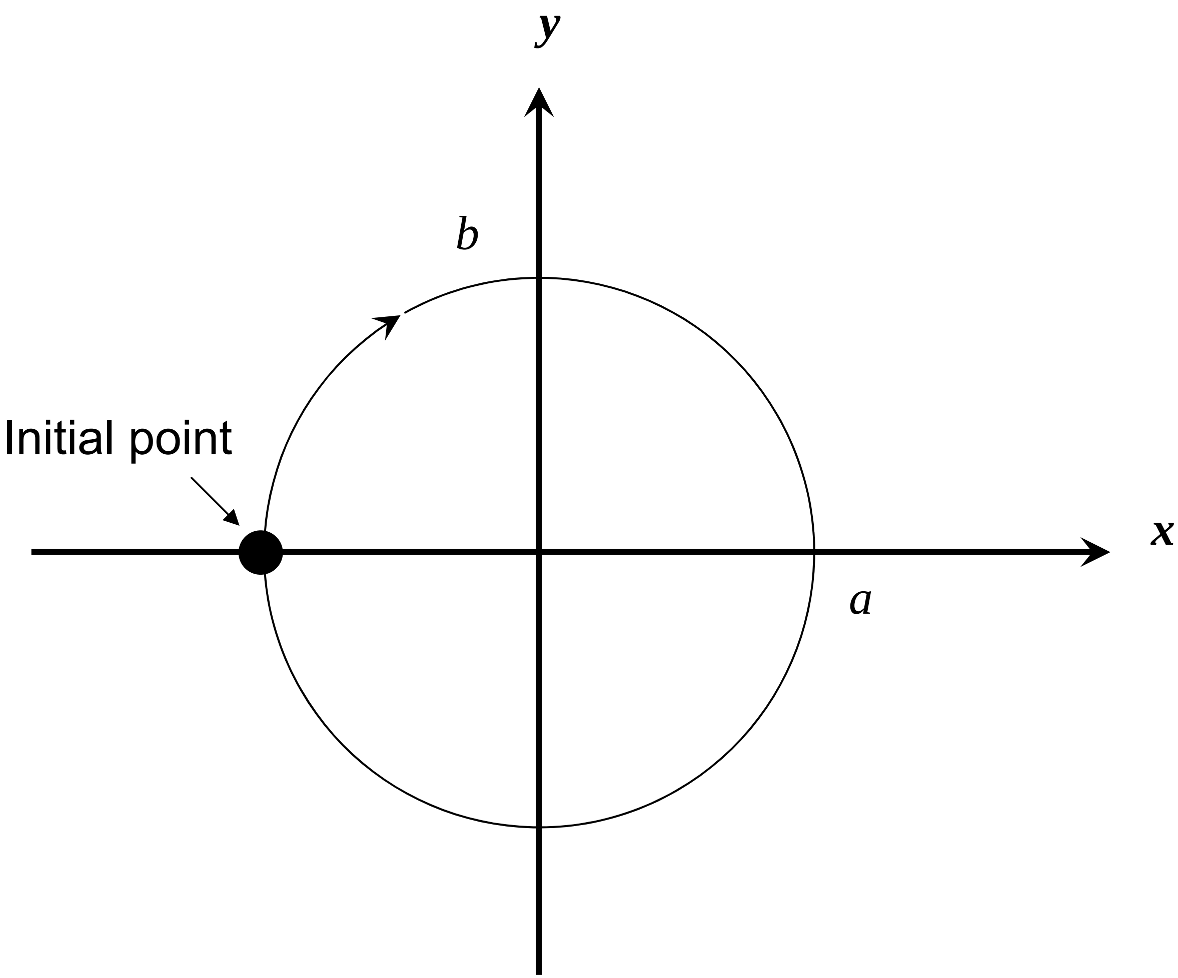


$$x = a\cos t$$

$$y = b\sin t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

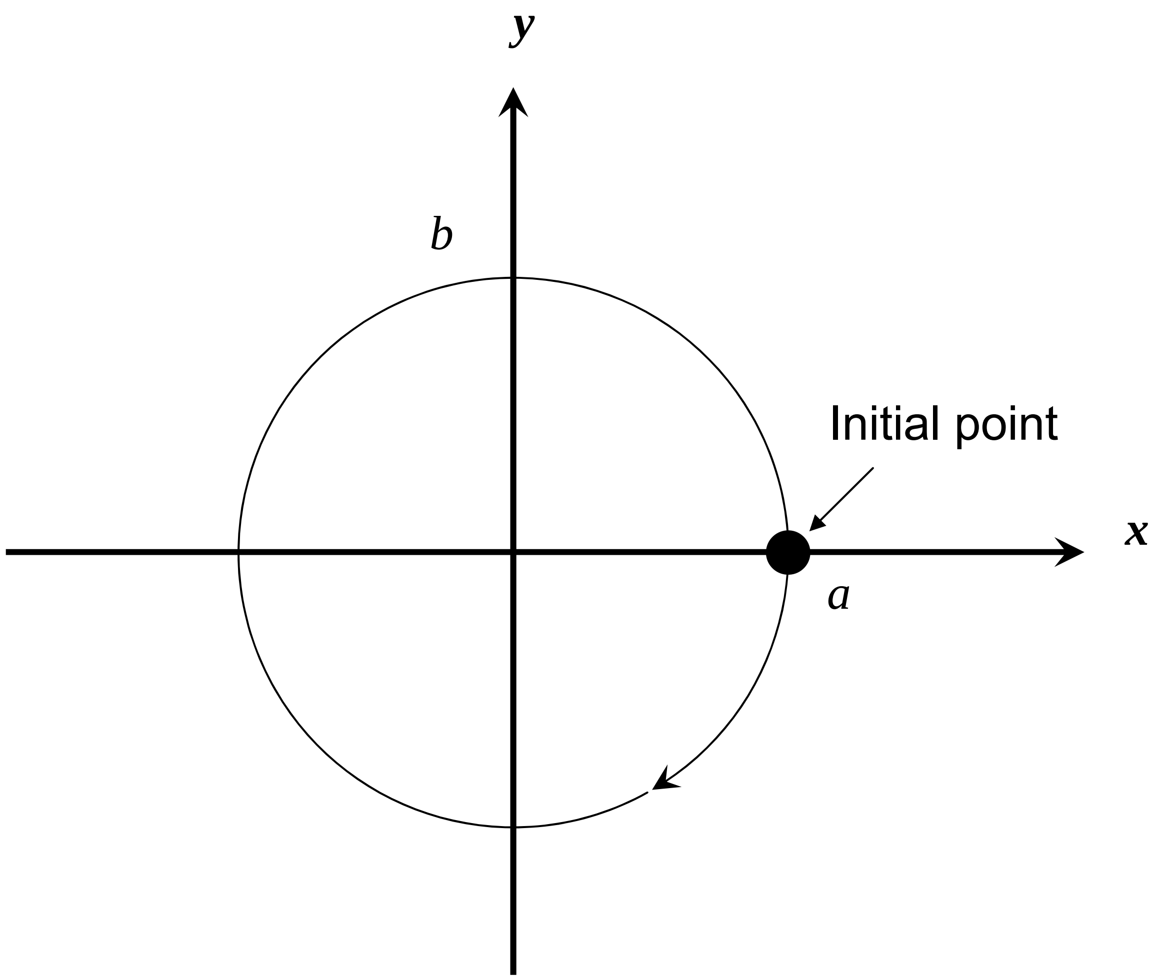


$$x = -a\cos t$$

$$y = b\sin t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

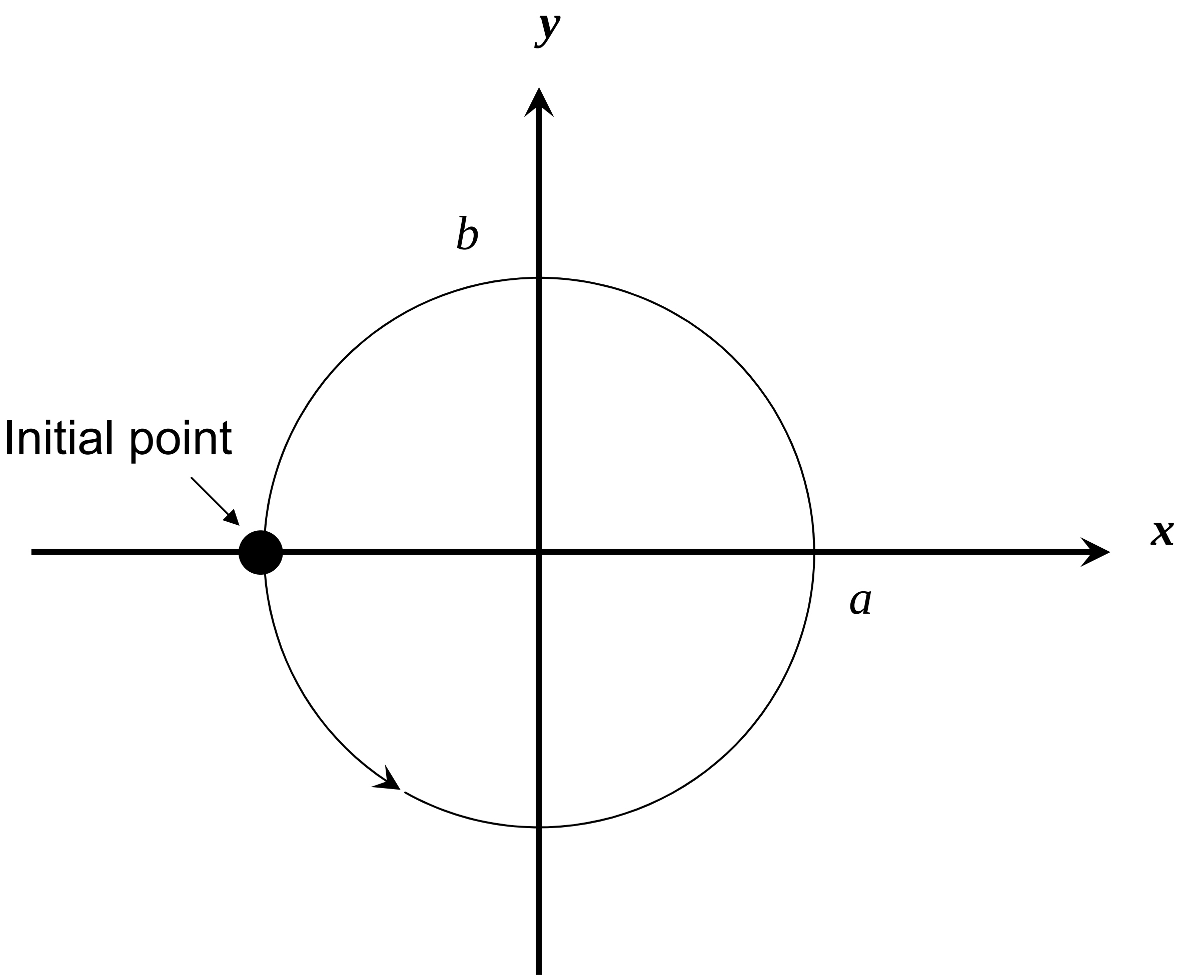


$$x = a\cos t$$

$$y = -b\sin t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

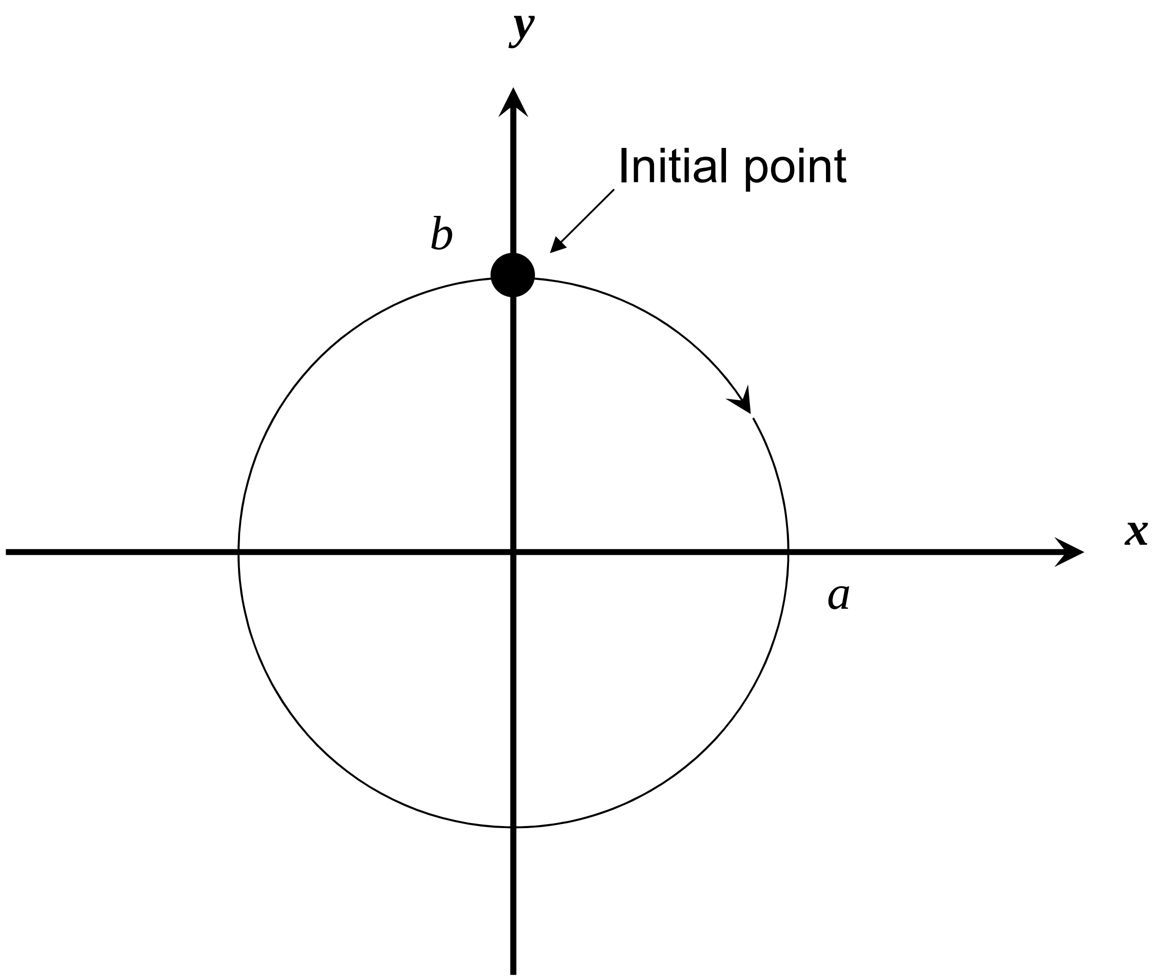


$$x = -a\cos t$$

$$y = -b\sin t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

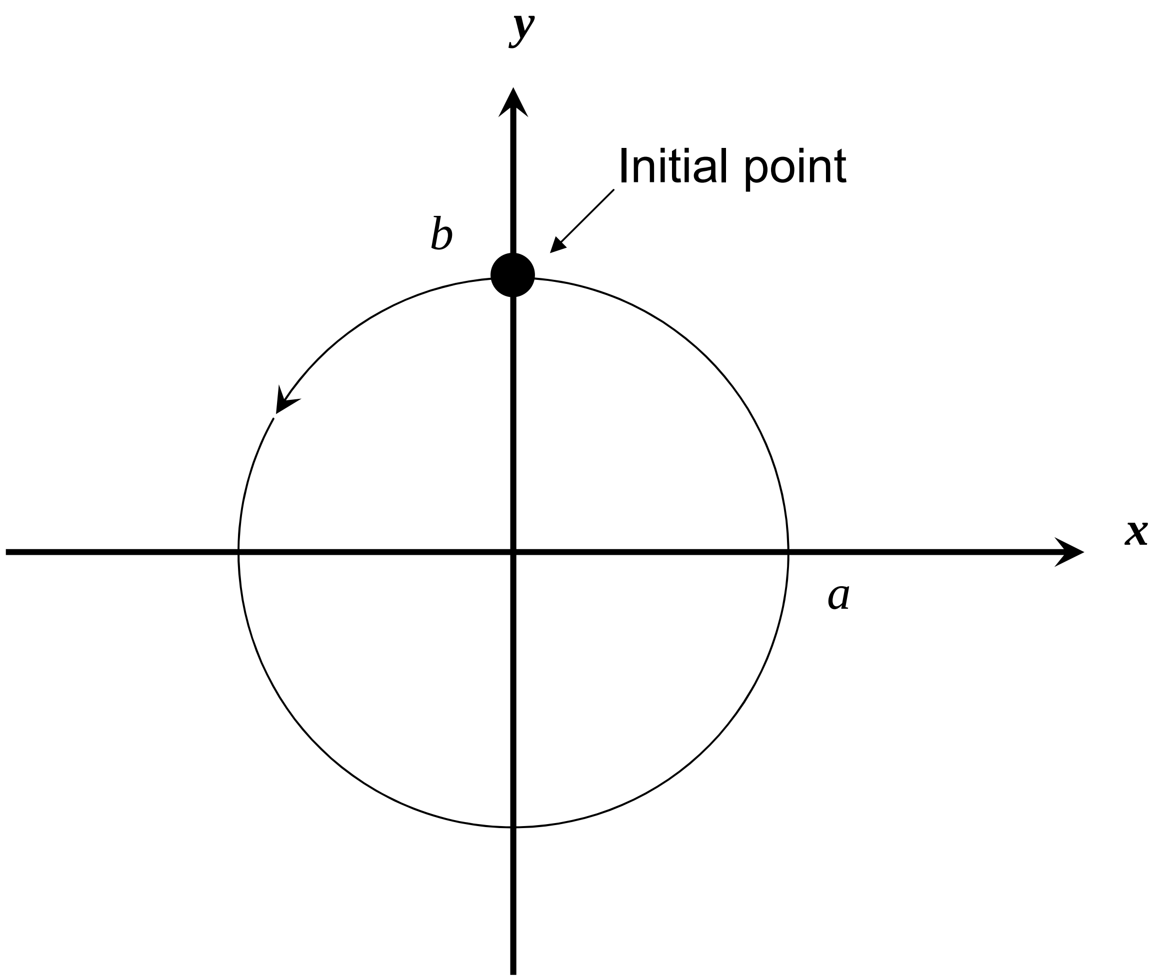


$$x = a\sin t$$

$$y = b\cos t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

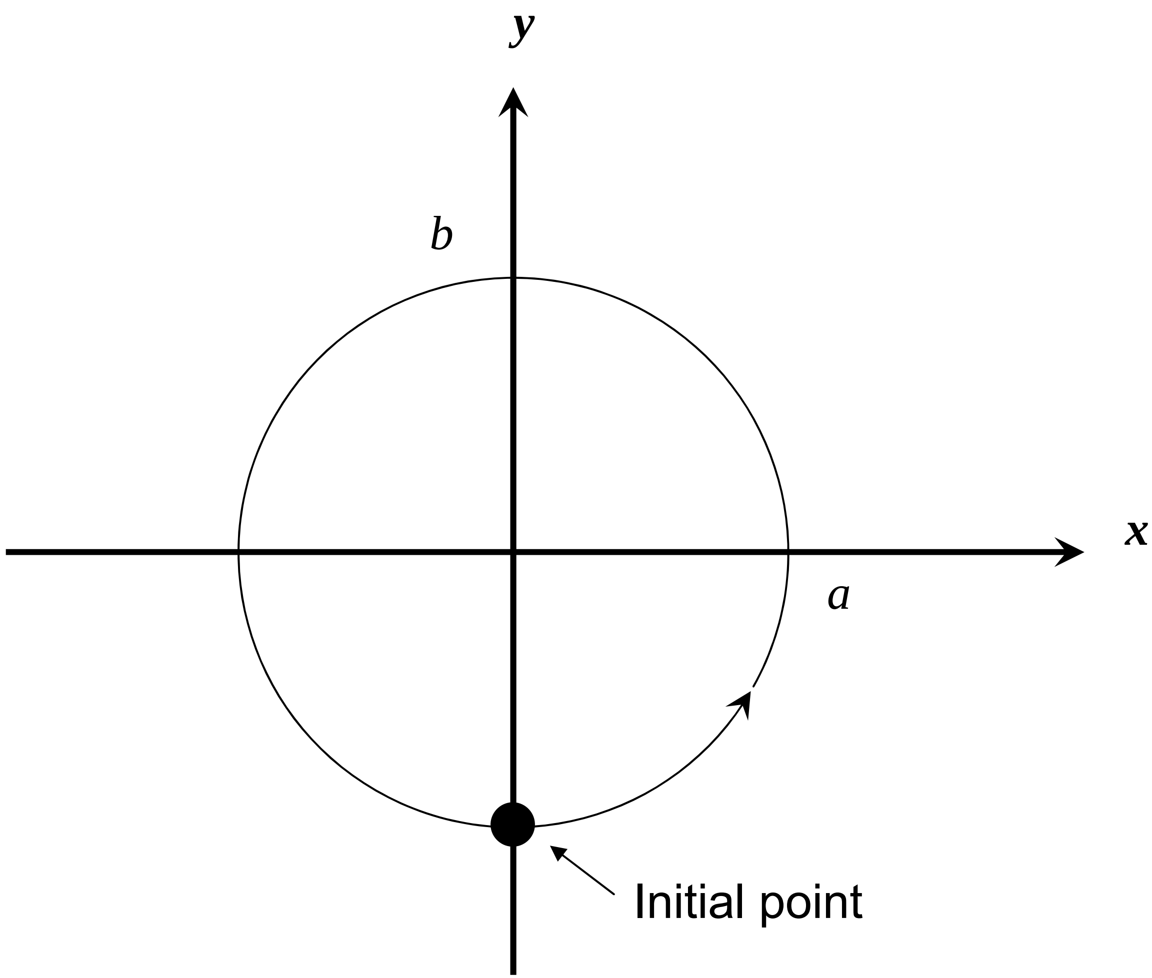


$$x = -a\sin t$$

$$y = b\cos t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)

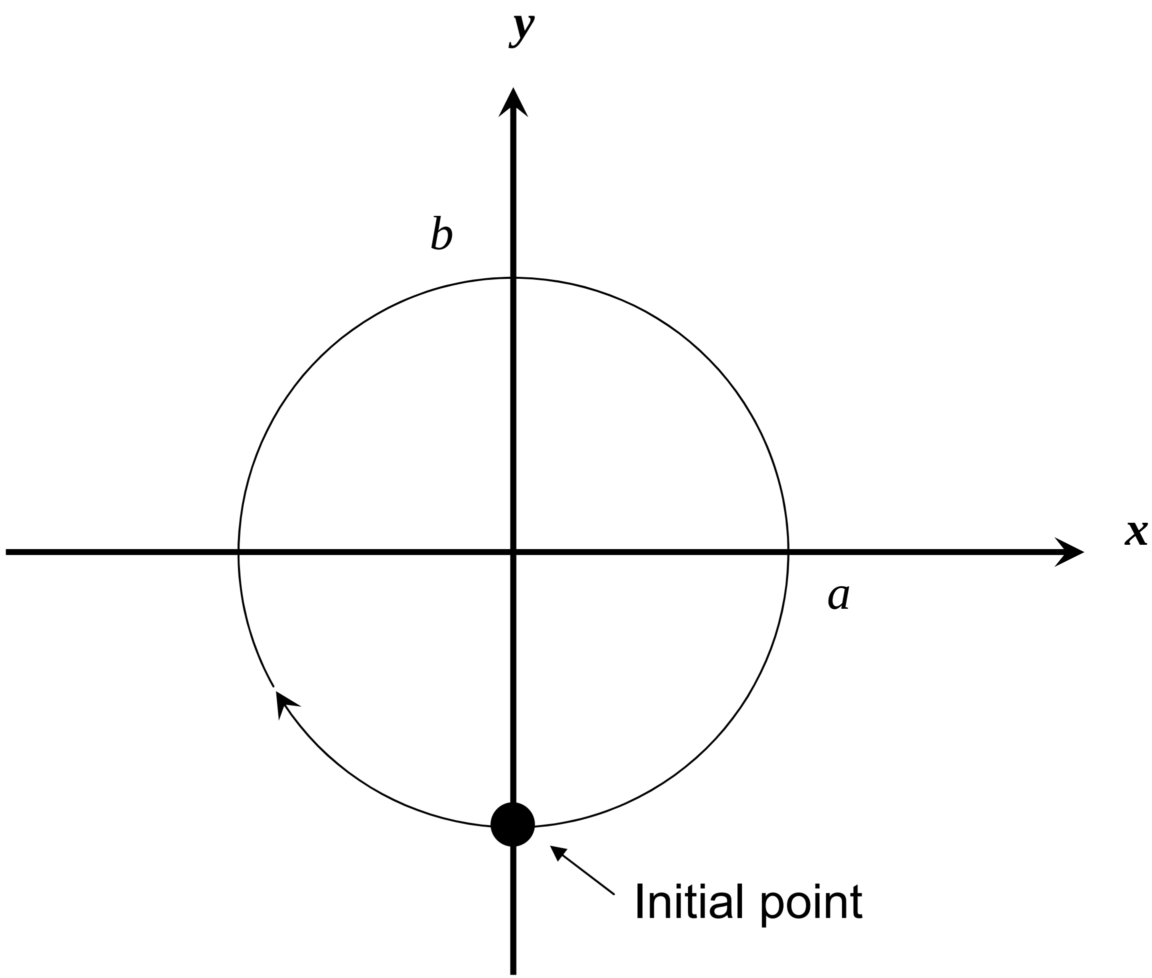


$$x = a \sin t$$

$$y = -b \cos t$$

$$0 \le t \le 2\pi$$

Parametric equation (circles & eclipses)



$$x = -a \sin t$$

$$y = -b \cos t$$

$$0 \le t \le 2\pi$$

Calculus of parametric equation

Differentiation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d(dy/dx)/dt}{dx/dt}$$

$$= \frac{dy'/dt}{dx/dt}$$

$$\left.\frac{dy}{dx}\right|_{x=c} = \text{tangent line at } x = c$$

$$\frac{d^2y}{dx^2} > 0 \qquad \text{concave up}$$

$$\frac{d^2y}{dx^2} < 0 \qquad \text{concave down}$$

Integration

$$I = \int_a^b y(x)dx$$

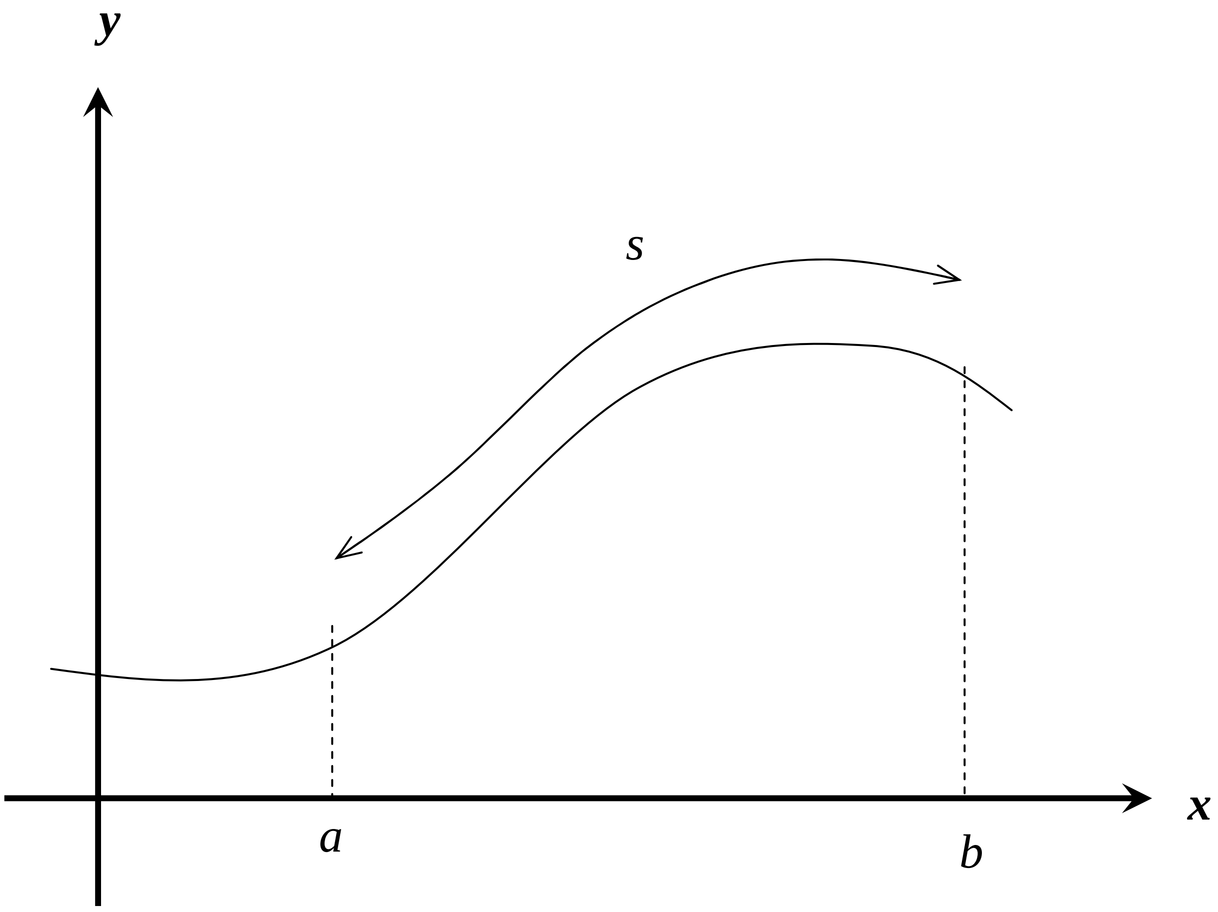
$$= \int_a^b y(x)dx\frac{dt}{dt}$$

$$= \int_{t_0}^{t} y(t)\frac{dx}{dt}dt$$

$$\therefore I = \int_{t_0}^{t} y(t)\dot{x}(t)dt$$

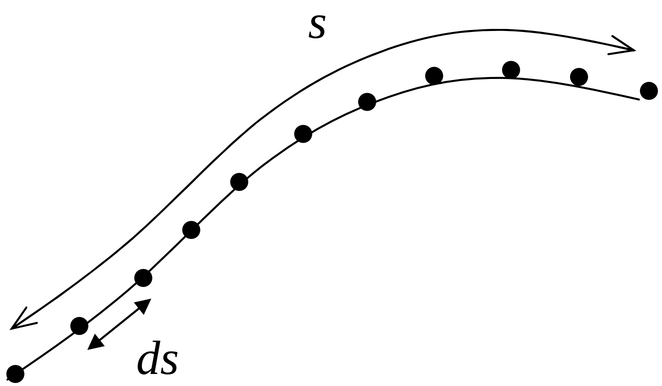
Curve length & surface area

How to find out the length of curve s ?



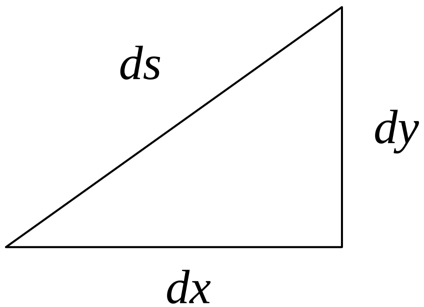
$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

First: *Divide s into very small portion, named ds*



Second: *Approximate ds as small line segment*

Third: *Use Pythagoras theorem after resolving ds into dx and dy*



$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dx^2 + dy^2$$

Since we are dealing with time / parametric equation dt must be substituted together with ds, dx & dy

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

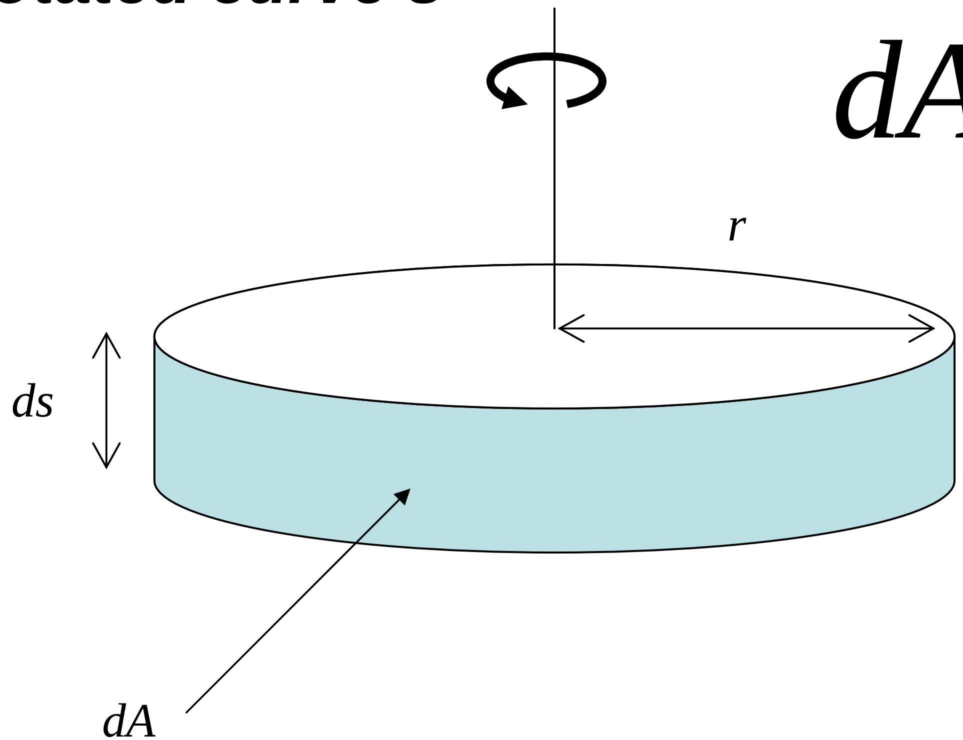
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

→ *Curve length equation*

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Finding surface area (shaded) for rotated curve s



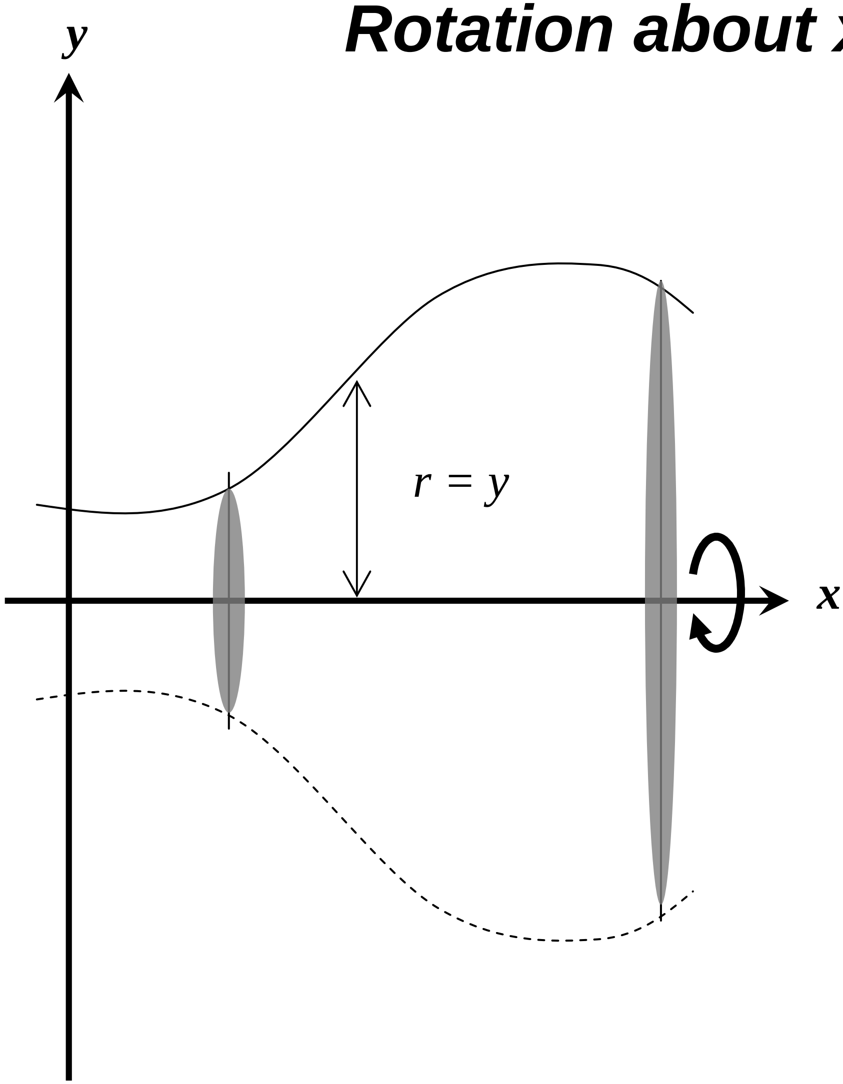
$$dA = 2\pi r\, ds$$

Shaded area = (circumference of circle) × (height)

Circumference of circle = 2*πr*

Height = *ds*

Rotation about x-axis @ y = 0

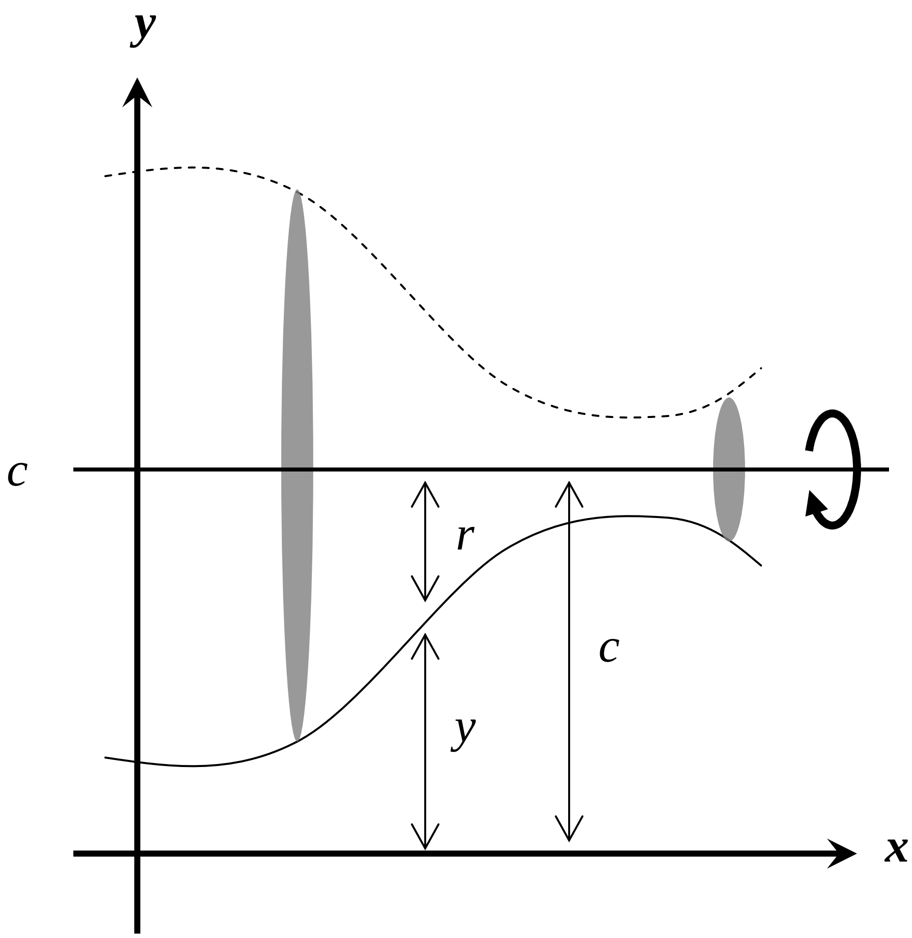


$$dA = 2\pi r\,ds$$

$$dA = 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$A = \int_{t_0}^{t} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Rotation about y = c (above the curve)



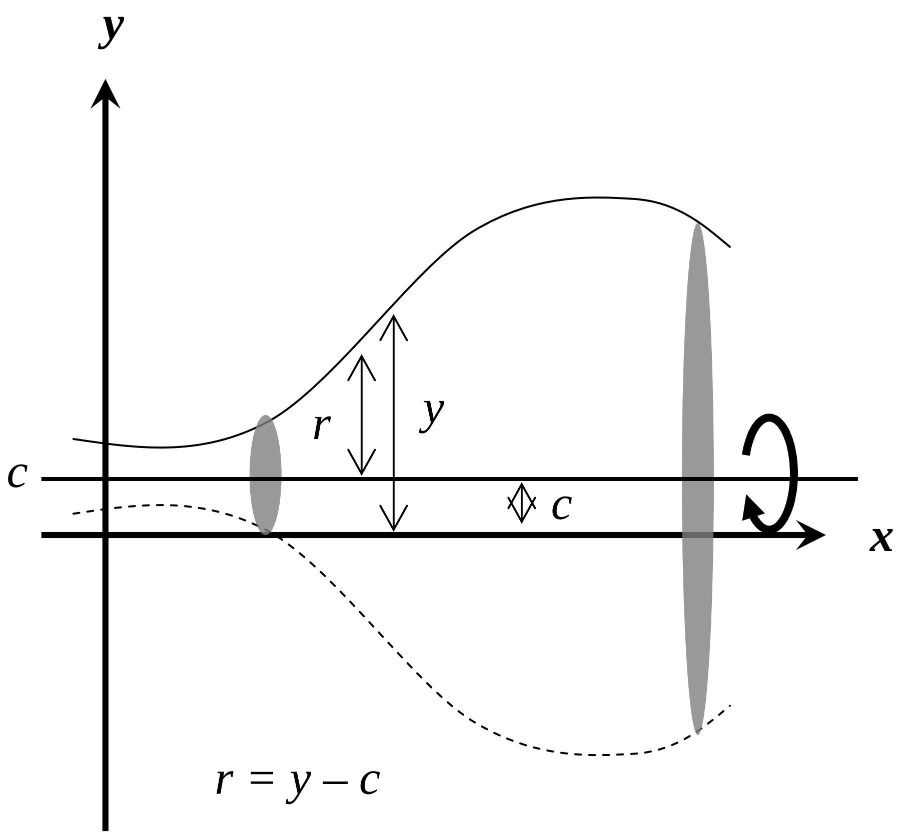
$r = c - y$

$$dA = 2\pi r\, ds$$

$$dA = 2\pi (c-y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$A = \int_{t_0}^{t} 2\pi (c-y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Rotation about y = c (under the curve) (1)

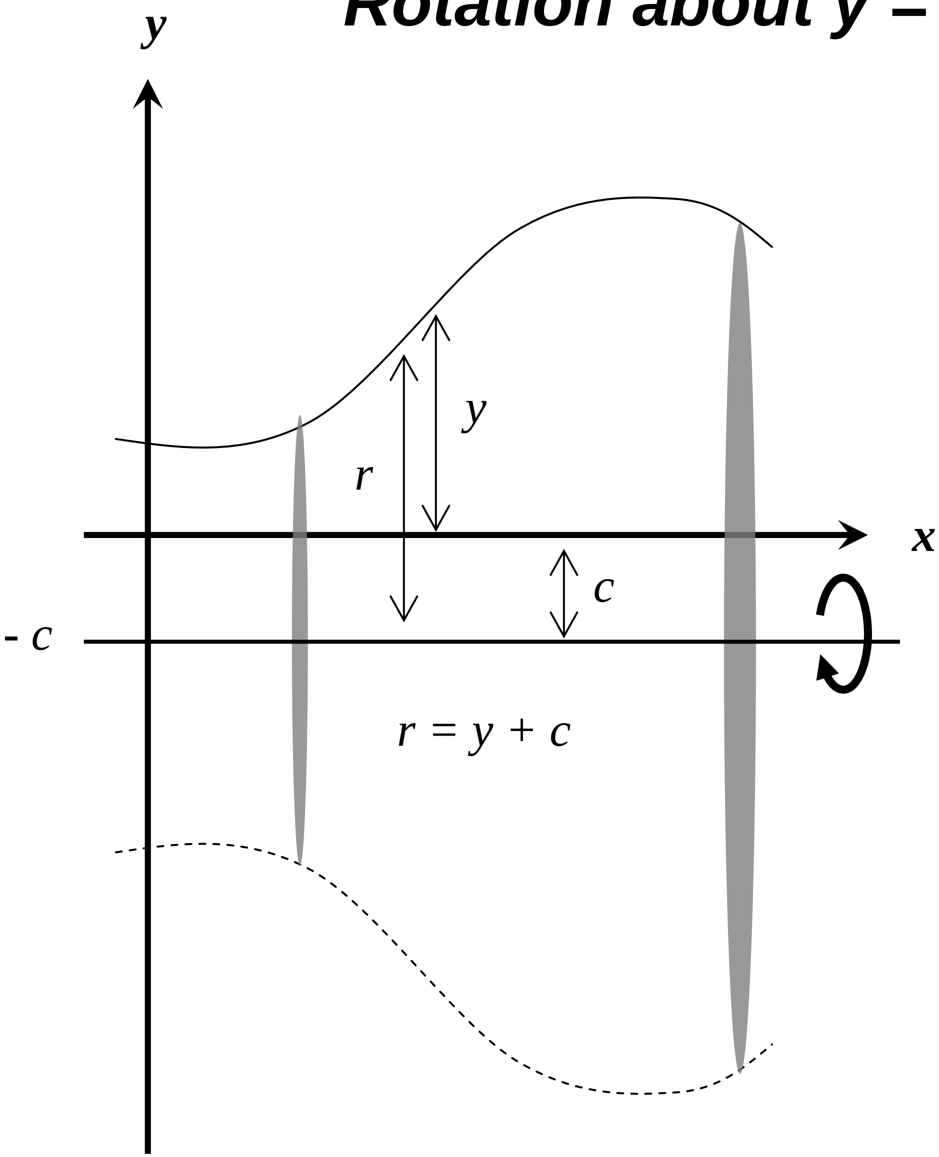


$$dA = 2\pi r\, ds$$

$$dA = 2\pi (y-c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$A = \int_{t_0}^{t} 2\pi (y-c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Rotation about y = -c (under the curve) (2)

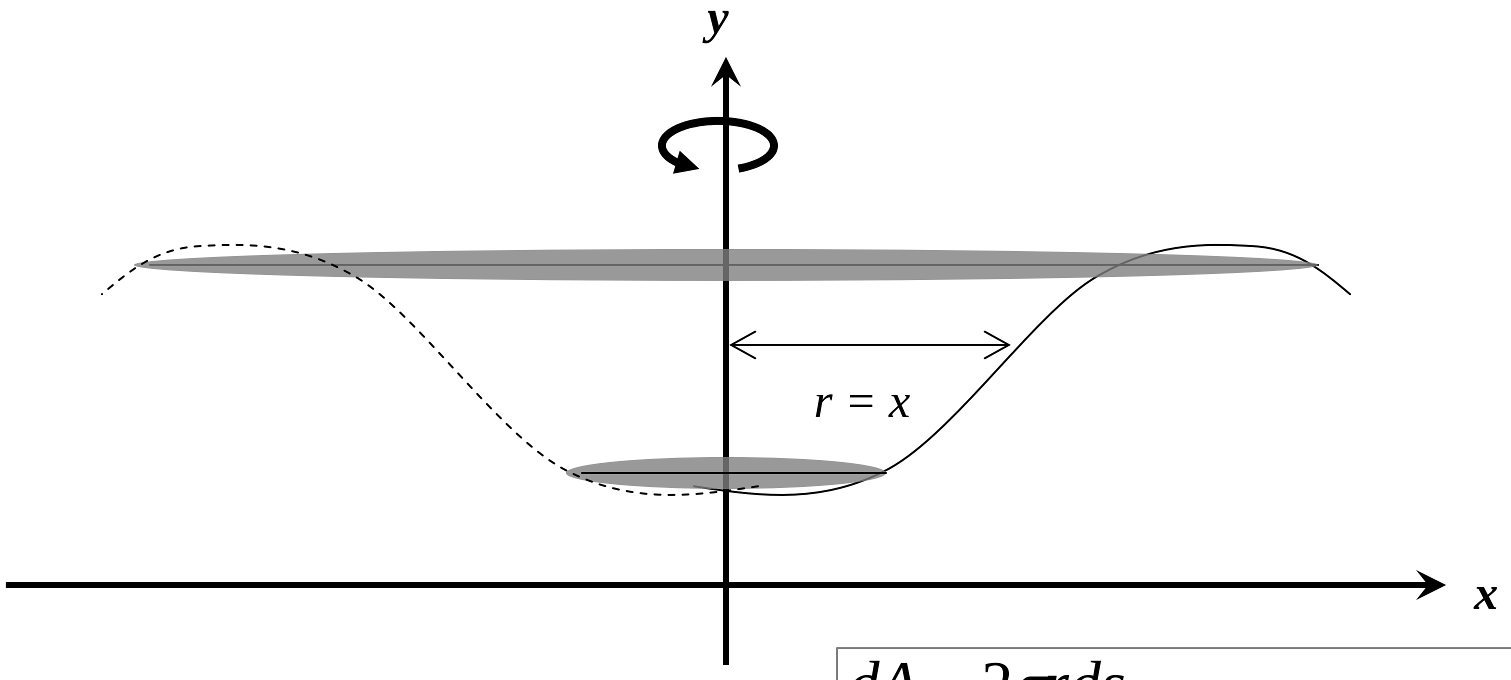


$$dA = 2\pi r\,ds$$

$$dA = 2\pi(c+y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$A = \int_{t_0}^{t} 2\pi(c+y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Rotation about y-axis @ x = 0

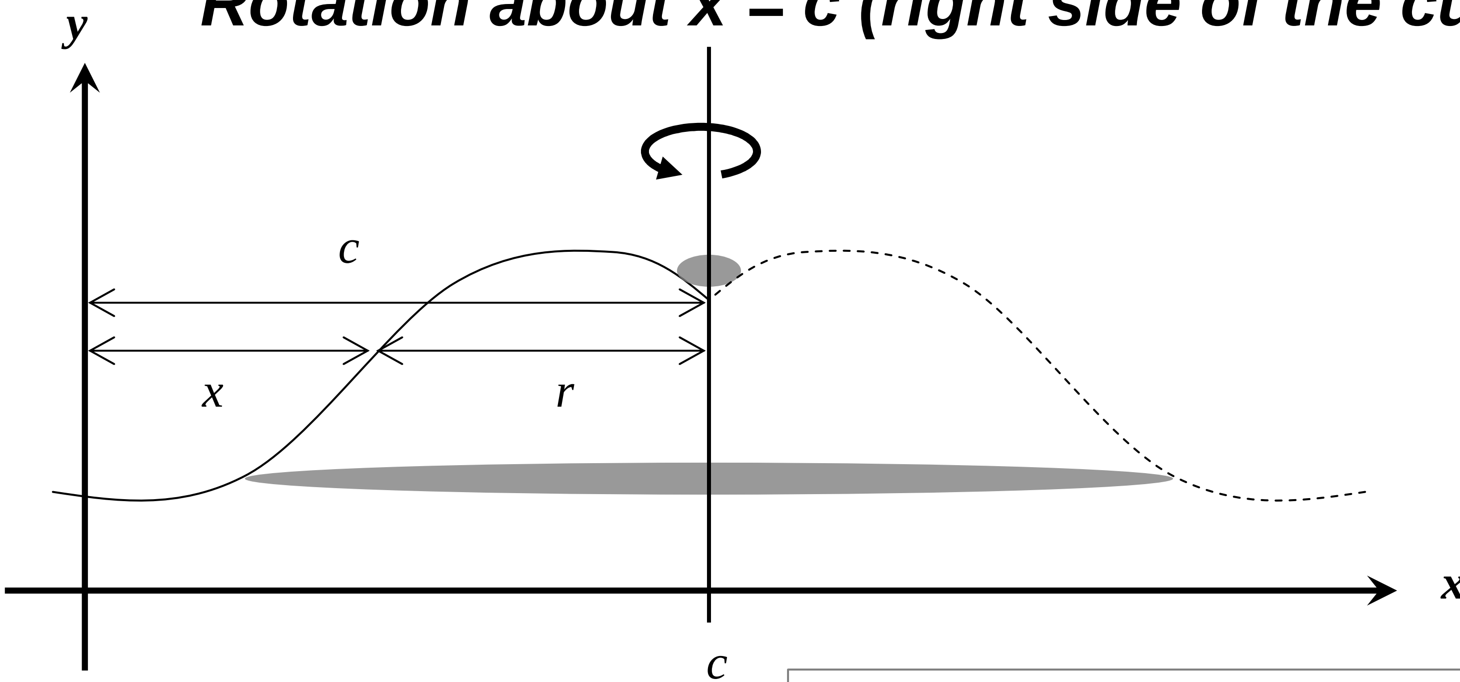


$$dA = 2\pi r ds$$

$$dA = 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$A = \int_{t_0}^{t} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Rotation about x = c (right side of the curve)



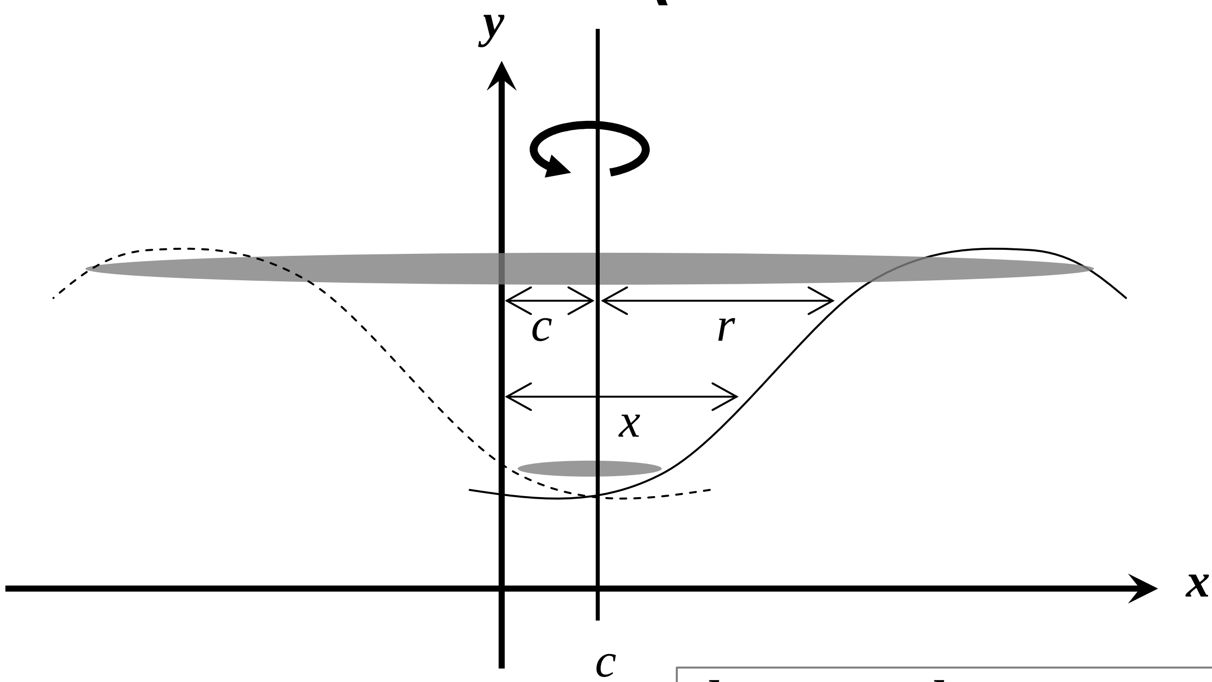
$r = c - x$

$$dA = 2\pi r\,ds$$

$$dA = 2\pi(c-x)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$A = \int_{t_0}^{t} 2\pi(c-x)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Rotation about x = c (left side of the curve) (1)



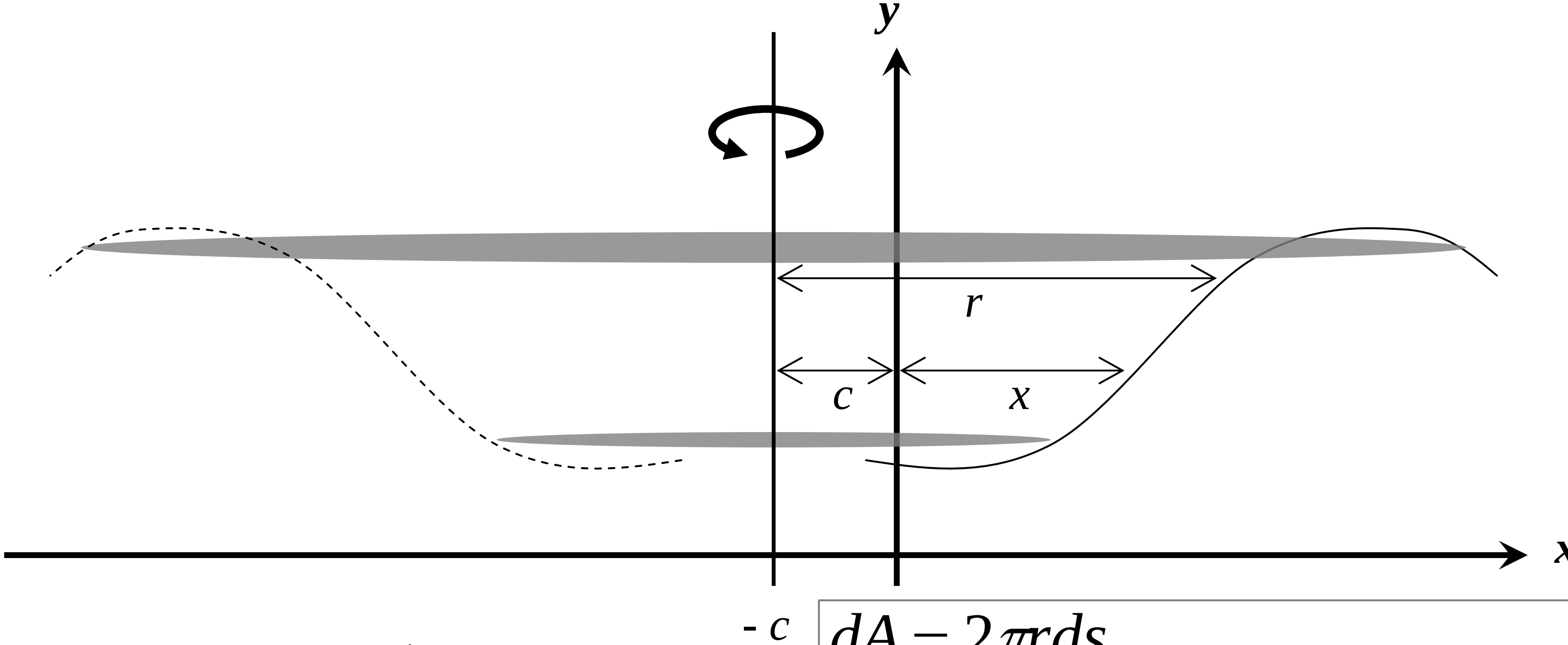
$r = x - c$

$$dA = 2\pi r\,ds$$

$$dA = 2\pi(x-c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$A = \int_{t_0}^{t} 2\pi(x-c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Rotation about x = -c (left side of the curve) (2)



$r = x + c$

$$dA = 2\pi r\,ds$$

$$dA = 2\pi(x+c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$A = \int_{t_0}^{t} 2\pi(x+c)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$