Thermodynamically Valid Noise Models for Nonlinear Devices

by

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Abstract

Noise has been a concern from the very beginning of signal processing and electrical engineering in general, although it was perhaps of less interest until vacuum-tube amplifiers made it audible just after 1900. Rigorous noise models for linear resistors were developed in 1927 by Nyquist and Johnson [1, 2]. However, the intervening years have not brought similarly well-established models for noise in nonlinear devices.

This thesis proposes using thermodynamic principles to determine whether a given nonlinear device noise model is physically valid. These tests are applied to several models. One conclusion is that the standard Gaussian noise models for nonlinear devices predict thermodynamically impossible circuit behavior: these models should be abandoned. But the nonlinear shot-noise model predicts thermodynamically acceptable behavior under a constraint derived here. This thesis shows how the thermodynamic requirements can be reduced to concise mathematical tests, involving no approximations, for the Gaussian and shot-noise models.

When the above-mentioned constraint is satisfied, the nonlinear shot-noise model specifies the current noise amplitude at each operating point from knowledge of the device $v - i$ curve alone. This relation between the dissipative behavior and the noise fluctuations is called, naturally enough, a fluctuation-dissipation relation. This thesis further investigates such FDRs, including one for linear resistors in nonlinear circuits that was previously unexplored.

The aim of this thesis is to provide thermodynamically solid foundations for noise models. It is hoped that hypothesized noise models developed to match experiment will be validated against the concise mathematical tests of this thesis. Finding a correct noise model will help circuit designers and physicists understand the actual processes causing the noise, and perhaps help them minimize the noise or its effect in the circuit.

Thesis Supervisor: John L. Wyatt, Jr.
Title: Professor of Electrical Engineering and Computer Science

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My advisor, Prof. John Wyatt, offered me this research opportunity over the phone when I was in Paris. In spite of our different work schedules, the retinal implant project that takes most of his time, and my year as President of the Graduate Student Council, we are both still enthusiastic about my work.

My committee also included Prof. Paul Penfield of MIT and Prof. Roger Brockett from Harvard. Prof. Penfield kept urging me to connect more to the physical reality and suggested Chapter 5; Prof. Brockett taught me a great deal of the math and worked to keep the rest honest, and Chapter 3 is the result.

Prof. Brian Anderson of the Australian National University posed the question answered in Chapter 6.

Prof. Jacob White's computers caused me hours of frustration, but they were a welcome relief from frustrations with my thesis. Prof. White also provided key recommendations for my summer job at Analog Devices and the Intel Fellowship.

Dr. Robert Reich hired me for a summer job at Lincoln Lab that gave me real experience. He also allowed me to use the equipment on my own time to perform a few experiments relating to my work. These are summarized in Appendix A.

Prof. John Tsitsiklis hired me as a teaching assistant for 6.262, discrete stochastic processes, even though I had not taken the class previously. I ended up with a much better understanding of Markov, renewal, and Poisson processes.

Thanks are also due to everyone who helped with papers or technical details along the way: Professors Jesus del Alamo, Hermann Haus, Alex Megretski, Sanjoy Mitter, Terry Orlando, Rahul Sarpeshkar, Steve Senturia, Jeff Shapiro, Dan Stroock, all of MIT; Prof. Erhan Çinlar of Princeton University; Prof. Wolfgang Mathis and Dr. Laurens Weiss of Otto-von-Guericke University, Magdeburg, Germany; Dr. Gerald Seidler and Dr. Ned Wingreen of NEC Research; and Prof. Carver Mead of Cal Tech.

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But it was my wife and best friend, Susan, who provided the most encouragement. She has been there for all of the past five years, the first five years of our marriage: sickness and health, somewhat poorer, but always better.

Contents

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List of Symbols

Variables:

ρ	probability density
ρ^o, ρ_{eq}	equilibrium probability density
ρ_{ss}	steady-state probability density
ρ_q	probability density for charge
ρ_v	probability density for voltage
p	discrete probability distribution
p^o	equilibrium probability distribution
q	charge
ϕ	flux
E	energy
v, V	voltage (time-varying or constant)
i, I	current (time-varying or constant)
$i = g(v)$	current through a (nonlinear) conductor
$v = f(q)$	voltage across a (nonlinear) capacitor
$i = h(\phi)$	current through a (nonlinear) inductor
$R_{ii}(t)$	autocorrelation (of current noise)
$S_{ii}(\omega)$	power spectral density (of current noise)
S_R	entropy of the thermal reservoir
S_{LC}	entropy of the inductor flux and capacitor charge distributions
$\xi(t)$	random process; generally unit-variance, white, and Gaussian
ζ	parameter for interpretation of stochastic integral
$w(t)$	Brownian motion
$W(x, y)$	transition rate from x to y
λ	rate of a Poisson counter

Constants:

e	electron charge, 1.602×10^{-19} C
k	Boltzmann's constant, 1.38×10^{-23} J/K
v_T	thermal voltage kT/e, 0.026 V at T=300 K

Parameters:

C	capacitance
G	conductance
L	inductance
R	resistance
B	bandwidth
T	absolute temperature, in Kelvin (K)
I_S	saturation current
n_{po}	electron concentration in the bulk p-region
p_{no}	hole concentration in the bulk n-region

Notation:

$\Pr\{\omega\}$	probability of an event y
$\overline{x}$ or $E\{x\}$	expectation of the random variable x
$\mathrm{Var}\{x\}$	variance of the random variable x
$\dot{x}$	time derivative of the variable x
$\mathbf{x}^T$	transpose of the vector $\mathbf{x}$
$\mathrm{tr}\{\mathbf{M}\}$	trace of the matrix $\mathbf{M}$
$\triangleq$	defined equal to
$\blacksquare$	end of proof

Chapter 1

Introduction

Noise has long been a concern in signal processing and electrical engineering in general. Schottky was investigating noise in diodes in 1918 [3]. The field of noise theory got its firmest foothold in 1927 with two fundamental papers by Johnson [2] and Nyquist [1]. Johnson's experimental measurements of noise in linear resistive materials motivated Nyquist to develop a theory to explain the noise from physical first principles. Noise continues to be of great concern, particularly in low power applications, or in channels where the communication bandwidth is approaching the channel capacity. Even digital logic in modern integrated circuits can be affected, as the supply voltage is decreased so that the difference between a "0" and a "1" is smaller.

The circuits of modern electrical engineering do not consist only of linear resistors, nor are these devices necessarily the most significant noise sources. Any dissipative device, that is, a device that converts electrical power into heat, also exhibits electrical noise. Devices such as diodes and transistors also need valid noise models.

This thesis will investigate thermodynamic requirements and restrictions for noise models for nonlinear devices. Emphasis in the electrical engineering world has tended to focus on models that accurately reproduce experimental measurements. However, if the predictions of a hypothesized model violate physical law, such as the laws of thermodynamics, it cannot be valid. Of course, if a valid model fails to predict

experiment, it is useless for design purposes. It is expected that obtaining a more complete model that satisfies thermodynamics and reproduces experiment will lead to a deeper understanding of the physical processes occurring in the device. With a deeper understanding of the noise process, designers will have more direct techniques for reducing the noise or its effect in circuits.

1.1 Noise Models

There are two primary noise models of interest for this thesis: Gaussian thermal noise and Poisson shot noise. Nyquist-Johnson or thermal noise is the simplest model for noise. Nyquist originally derived the model for linear resistors at equilibrium to explain Johnson's experimental results. A resistor of value R will have a voltage power spectrum that is white and has magnitude $2kTR$ (the original works used the value $4kTR$, because the authors only allowed positive frequencies). The voltage waveform will approximate the fictional Gaussian white noise process.

A Gaussian noise model is one for which the noise voltage (or current) at any particular instant is probabilistically selected according to a Gaussian distribution. Commonly, the noise model is also assumed to have a flat power spectral density (at least out to the frequencies of interest for the circuit); this type of model is called "white noise." (A correlation between time instants would produce "colored noise." The voltage would still be picked from a Gaussian distribution, but the mean or variance would depend on past voltages.) The power spectral density is the Fourier transform of the autocorrelation (two-time correlation function). For a white noise process, the inverse Fourier transform yields a delta-function, meaning that the signal at any time is uncorrelated with the signal at any other time.

It is the Gaussian nature of the random process that makes this model so mathematically interesting; statistical quantities are easy to compute. A Gaussian random variable is completely defined by its first two moments, the mean and variance. The

mathematical tractability of Gaussian white noise has encouraged engineers to extend this theory to nonlinear resistors. The mean value is assumed to be zero (this is in fact one of the requirements we will present later). The variance is generally related in some fashion to the incremental resistance, to align with the Nyquist formula. For example, Gupta [4] proposes the formula

$$E\{v_n^2\} = \mathrm{Var}\{v_n\} = 4kTB \left(\frac{dV}{dI} + \frac{1}{2}I\frac{d^2V}{dI^2} \right)\bigg|_{I=I_{dc}},$$

where k is Boltzmann's constant, T the Kelvin temperature, and B the bandwidth of measurement. Unfortunately, such extensions to nonlinear resistors are physically wrong. This particular formula would predict a negative mean square voltage fluctuation for the tunnel diode in the region where the device has a negative incremental resistance. This thesis proves that every white Gaussian noise model for nonlinear devices violates thermodynamic principles.

The second type of noise considered in this paper is shot noise. The term shot noise [3, 5, 6] dates back to early work by Schottky on vacuum diodes and triodes. Since current is carried by discrete particles with discrete arrival times, the current will not be uniform. The effect of random arrival times of electrons is named shot noise. A Poisson point process can be used as the mathematical model behind shot noise. The shot noise model will also have a white power spectral density for current at equilibrium.

For a Poisson noise model, the current at any instant is either zero or a delta-function of strength corresponding to a single electron. Arrivals of a Poisson process are conditionally independent given the rate; this lack of correlation between the current at different instants leads to a white power spectral density without us specifying that separately (as we needed to do for Gaussian noise).

Unfortunately, there is even confusion about shot-noise models. The *Art of Electronics* [7], which is generally a great guide for practical circuit design, has the state-

ment that shot noise "like resistor Johnson noise, is Gaussian and white."

One author [8] claims that any stationary, independent-increments process must be a combination of a Gaussian process (or Brownian motion) and a Poisson process. This is not entirely true, since a process with randomly-selected jump heights but Poisson arrivals would also fit the description. It is important to realize that the power spectral density determines correlation between two values of the process, which is separate from the distribution of the values themselves. Other random processes may make suitable noise models; the methods of this thesis are generally applicable and may help to determine which noise models ones are physically possible and justifiable. In particular, so-called $1/f$ noise, whose power spectral density falls off as $1/f$ for large frequencies, is lacking a rigorous model.

Both white noise (with infinite bandwidth) and $1/f$ noise are unphysical, because the mean-square value or energy of the signal is infinite. White noise is usually assumed to roll off above a certain frequency, because of the lowpass filter effect of parasitic capacitance and the intrinsic resistance. Further, from an experimental point of view, one is usually only interested in the response up to a given frequency. $1/f$ noise has a singularity at zero frequency, which requires a more delicate treatment so that the noise properties do not depend on peculiar quantities such as the age of the universe (the zero-frequency point corresponds to infinite time). We are unaware of a well-grounded theory giving the proper low-frequency behavior and the frequency at which the transition from $1/f$ occurs. Many noise models, whether for $1/f$ noise or other types of noise in nonlinear devices, are constructed to match experiment.

1.2   Thermodynamic Tests

There are four thermodynamic tests presented in this thesis to assess the validity of noise models. Chapter 2 applies the first three tests to a circuit with only an ideal capacitor in addition to the noise model, which makes for mathematically very simple

tests. This thesis goes on in Chapter 4 to check the noise models in more complicated circuits, with the goal of verifying correct behavior for arbitrary networks of lossless elements. Such checks were made for linear resistors in [9], which extends the Nyquist-Johnson model away from equilibrium. The fourth thermodynamic test is presented in Chapter 5.

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

One elementary consequence of the second law of thermodynamics is that no isothermal system can have as its sole effect the conversion of some amount of heat into work [10]. A noisy dissipative device at a fixed temperature T, biased at a voltage V with the resulting average current $\overline{i_T(V)}$, must not supply power, on average, to the external circuit, *i.e.*,

$$\overline{i_T(V)}\, V \geq 0, \quad \text{for } T > 0 \text{ and all } V.$$

Since the average current is assumed to be a continuous function of the applied voltage, this also implies that the average short-circuit current for a dissipative device must be zero.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

For any lossless lumped network in thermal equilibrium with a dissipative device at constant temperature, the equilibrium distribution for inductor fluxes $\boldsymbol{\phi}$ and capacitor charges $\mathbf{q}$ must have the Gibbs (or Maxwell-Boltzmann) form [11],

$$\rho^o(\boldsymbol{\phi}, \mathbf{q}) = A \exp\left[-E(\boldsymbol{\phi}, \mathbf{q})/kT\right], \tag{1.1}$$

where $E(\boldsymbol{\phi}, \mathbf{q})$ is the sum of all inductor and capacitor stored energies and A serves to normalize the distribution.

Thermodynamic Requirement #3: Increasing Entropy During Transients

The second law of thermodynamics must be satisfied during nonequilibrium transient behavior of any circuit driven by the fluctuations of the dissipative device. The total entropy of a circuit, *i.e.*, the sum of the entropies of the lossless elements and the reservoir, must be a nondecreasing function of time, with a maximum value corresponding to the equilibrium distribution [10].

Thermodynamic Requirement #4: No Heat Transfer between Two Devices at the Same Temperature

For any circuit consisting of two or more noisy devices, each in thermal contact with a thermal reservoir of a single temperature T, and any lossless lumped network, there should be no heat transfer between the devices, that is, no net power delivered or absorbed by any one of the devices. In contrast, heat should flow from the hotter to the cooler if the devices are in thermal contact with reservoirs at different temperatures, but the rate of flow will depend on specifics of the devices and the lossless network.

1.3   Mathematical Tools

This thesis is very mathematical. Since thermodynamic principles are based on probabilistic descriptions of systems, the reader is presumed to have a solid understanding of probability. Those readers unfamiliar with Gaussian and Poisson random variables are referred to Papoulis and Gallager [12, 13]. A working knowledge of measure theory [14] is also useful, particularly for the more mathematical treatments of [15, 16].

The reader should also understand the concepts of autocorrelation and power spectral density [17, 18]. The principle processes considered in this thesis are white noise processes, those random processes whose power spectral density is flat.

The stochastic calculations in this thesis make use of the Fokker-Planck equa-

tion (FPE) and the Master Equation, both versions of the "differential Chapman-Kolmogorov equation" [19]. The basic idea is that, for a differential equation driven by random processes, the sample-path solutions are not particularly interesting or useful. Instead, one looks for the evolution of the probability distribution for the system. The FPE, Master Equation, and Chapman-Kolmogorov equation are all forward evolution equations for the probability distribution of a system.

The FPE allows us to describe the behavior of a circuit driven by Nyquist-Johnson noise from a linear resistor. This topic was exhaustively developed in [9]. This thesis will expand the applications to nonlinear resistors.

The Master Equation is easier to understand conceptually. A good introduction, with excellent physical intuition, is found in van Kampen's book [20]. If the probability of being at a point y at time t is $P(y,t)$ and the rate of transition from x to y is $W(x,y)$, then the following Master Equation can be seen to describe the time evolution of the probability:

$$\frac{dP(y,t)}{dt} = \int \left[W(y-r,y)P(y-r,t) - W(y,y-r)P(y,t) \right] dr \qquad (1.2)$$

The first term describes flow into the point y from all other points $y - r$; the second term describes flow out of y.

The Master Equation is particularly useful for describing the behavior of a system which moves in random jumps. Examples include birth-death processes (an integer number of individuals are born or die at each step) or a random walk (each step of fixed length, randomly chosen left or right). This thesis uses the Master Equation to describe systems driven by Poisson processes.

The Fokker-Planck equation can be derived as a limit of the Master Equation. This derivation, as well as a second using a dual-space argument, will be presented in Chapter 3.

1.4   Fluctuation-Dissipation Theorems

This thesis is primarily concerned with the physical relations known as fluctuation-dissipation theorems. The Einstein relation is the earliest such theorem:

$$\frac{D}{\mu} = \frac{kT}{e},$$

where D is the diffusivity, μ the mobility, k Boltzmann's constant, T the temperature, and e the charge on an electron. The random diffusion of particles is related to their motion under an applied electric field, when they dissipate the applied power. Nyquist's theorem is also a fluctuation-dissipation theorem:

$$\mathrm{Var}\{v_n\} = 4kTBR.$$

In developing the shot-noise model, we will derive a nonlinear fluctuation-dissipation theorem. Stratonovich has done extensive work on nonlinear fluctuation-dissipation relations in [21], but the book is difficult to read, perhaps due to a poor translation.

Chapter 6 also investigates a fluctuation-dissipation theorem for a circuit containing linear resistors, but nonlinear energy storage elements. It is well known that the thermal noise behavior at the terminals of any linear time-invariant (LTI) RLC circuit can be predicted from knowledge of the driving-point impedance and temperature alone. This chapter examines the conjecture that similar results hold if the capacitors and inductors are nonlinear. We refine the conjecture by analyzing the behavior of an RLC bridge circuit with the nonlinear inductor and capacitor carefully matched so the terminal behavior reduces to that of a linear resistor R. We show that the terminal noise current is *not* that predicted by the Nyquist-Johnson model for R if the driving voltage is time-dependent or the inductor and capacitor are time-varying. This counterexample disproves the conjecture, which does hold, however, for the bridge circuit with nonlinear (but time-invariant) devices if the driving

voltage is zero or constant. The chapter makes exact calculations using techniques from stochastic differential equations and using reversibility arguments.

1.5 Contributions of this Thesis

In addition to presenting the thermodynamic requirements for noise models and reducing them to mathematical tests, this thesis presents the first nonlinear device noise model that satisfies all of these tests. Previous noise models have been derived experimentally, and only one or two of the thermodynamic requirements have been checked, if any.

This thesis also extends the fluctuation-dissipation theorem to include a specific circuit with nonlinear energy storage elements in steady-state. It then provides a counterexample for a further extension of the fluctuation-dissipation theorem for linear resistors but nonlinear energy storage elements in a specific nonequilibrium situation.

Many sections of this thesis have been accomplished as joint work with other researchers.

Chapter 2 recounts joint work with Professor Wyatt. After developing this model, we discovered a distinct but related treatment in [21], which is based on a complicated "kinetic potential" argument. It uses an approximation [21, eq. (3.3.43)], not used or needed here and handles the discontinuities in $v(t)$ differently. In addition, I have explicitly verified that the Poisson model satisfies the increasing entropy (using a new proof not found in our original paper) and heat transfer requirements.

Chapter 3 is a recapitulation of useful mathematical results from various sources. I have developed certain special cases that do not appear in the literature in order to address specific questions in this thesis.

Chapters 4 and 5 are, to my knowledge, completely new. The derivations are my work alone. This material expands the tests applied to the Poisson model (though

the necessary condition on the Poisson model is the same as in Chapter 2), and the mathematics are substantially more complicated than Chapter 2. The Poisson model for a nonlinear device could not be considered complete without the generalization beyond the single capacitor. Chapter 5 returns to the single capacitor case, but considers a further thermodynamic test that we had not previously applied to the Poisson noise model.

Chapter 6 consists of joint work with Prof. Anderson and Prof. Wyatt. Prof. Anderson had been thinking about the problem for many years and derived a nonlinear matching condition. I independently derived a different nonlinear matching condition and used it to derive the equilibrium and steady-state results for the noisy nonlinear circuit. The time-varying analysis was Prof. Wyatt's work.

The original dissertation was submitted May 19, 2000 to the Department of Electrical Engineering and Computer Science. This is a revised version, in which I have corrected minor typographical errors, which do not affect any of the conclusions. I used pdfTEXto generate PDF directly; spacing, linebreaks, and pagination may differ from the original, which used LATEXto generate PostScript.

Chapter 2

Nonlinear Device Noise Models: Satisfying the Thermodynamic Requirements

The material of this chapter appeared previously as "Nonlinear Device Noise Models: Satisfying the Thermodynamic Requirements," in *IEEE Trans. Electron Devices*, January 1999 [22]. A new proof of increasing entropy has been added, and several minor changes have been made to integrate this paper into the thesis.

2.1 Introduction

Unlike idealized capacitors and inductors, dissipative devices such as resistors, diodes, and transistors degrade electrical energy to thermal energy. This thermal energy is expressed as electrical noise.

The Nyquist-Johnson thermal noise model asserts that the behavior of a linear conductor G at thermal equilibrium at a temperature T Kelvin is accurately modeled by the Norton representation in Fig. 2-1, where (ignoring the high-frequency roll-off in the infrared) the current noise source is zero-mean and white with power spectral
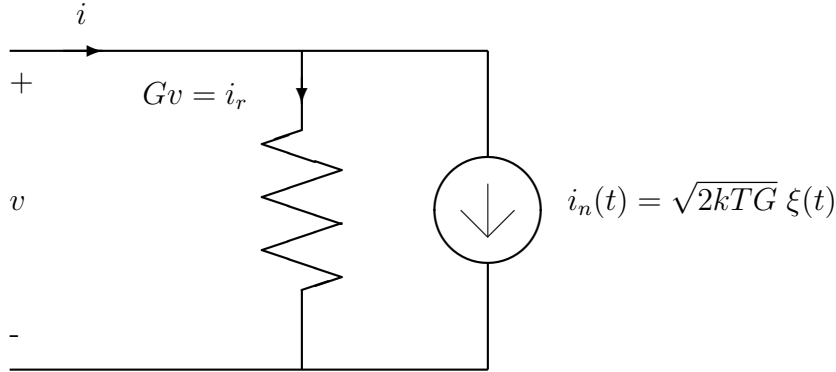
Figure 2-1: The Norton equivalent Nyquist-Johnson noise model for a linear conductor.

density[1]

$$S_{ii}(\omega; T) = 2kTG, \tag{2.1}$$

independent of ω. Equation (2.1) involves only the conductance and the temperature: it is independent of the physical construction of the conductor [1, 2]. Nyquist's theoretical derivation was based on fundamental thermodynamic principles.

The aptly-named fluctuation-dissipation theorem [23, 24, 25, 26] governs the noisy fluctuations in macroscopic variables of dissipative systems. It generalizes Johnson's and Nyquist's resistor noise model to mechanical, chemical, hydraulic, and other domains. But the classical fluctuation-dissipation theorem is limited to linear dissipative elements. This chapter will show how thermodynamics also constrains the behavior of nonlinear devices.

The physical idea in this chapter is similar to that in [1], where resistors were connected to a transmission line. In Fig. 2-2, a nonlinear 2-terminal device at constant temperature is connected to a fairly arbitrary lossless network,[2] which contains, in general, nonlinear multi-terminal inductors and capacitors plus ideal gyrators and

---

[1]The power spectral density expression is $4kTG$ when only positive frequencies are considered.

[2]The network cannot contain ideal diodes, those whose constitutive relations lie on the $v - i$ axes. Chapter 4 considers this arbitrariness, allowing nonlinear but reciprocal energy storage elements and linear but possibly nonreciprocal interconnections, such as gyrators.
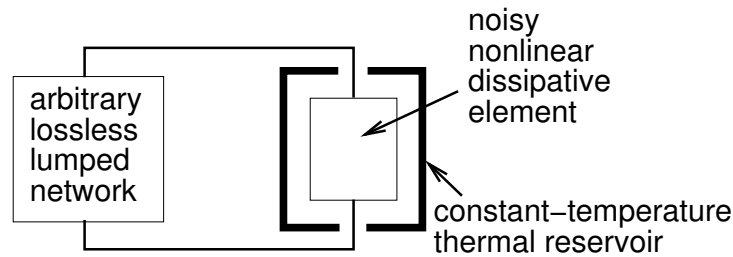
Figure 2-2: Test circuit with nonlinear device in thermal equilibrium with an isothermal reservoir, connected to a lossless network.

transformers, as in [9], though the results in this chapter are obtained with a single linear capacitor.

At thermal equilibrium, the voltage and current fluctuations are generally small and the nonlinear device behavior could be approximated by linearizing about the origin of the $v - i$ curve. But on rare occasions, the fluctuations will be large enough to briefly drive the device into the nonlinear regime. *Its behavior during such large equilibrium fluctuations is also constrained by thermodynamic principles.* This requirement serves as a pruning mechanism for rejecting many noise models *ab initio* and tentatively accepting others: models that predict non-thermodynamic behavior during large fluctuations (however rare) are *non-physical* and should be abandoned.

This chapter will introduce three equilibrium and nonequilibrium requirements that greatly restrict and simplify the class of acceptable models. These requirements will be presented as simple mathematical and circuit-theoretical tests for some noise models. In the literature on nonlinear noise modeling, approximations and assumptions often introduce confusion over the domain where results apply. This chapter treats the nonlinear problems *exactly*, using stochastic differential equation and Master Equation methods (but we restrict consideration to two-terminal, voltage-controlled resistive elements for simplicity.) It turns out that consistency with thermodynamics cannot be determined from a model's noise spectrum alone, but depends

critically on further statistical details.

Section 2.2 lists specific tests a model must pass. Sections 2.3 and 2.4 introduce the Gaussian noise model for linear and nonlinear elements. Section 2.5 develops the shot-noise model and can be read independently of Sections 2.3 and 2.4. Section 2.6 compares the two acceptable models.

2.2 Thermodynamic Requirements on Resistor Noise Models

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

One elementary consequence of the second law of thermodynamics is that no isothermal system can have as its sole effect the conversion of some amount of heat into work [10]. A noisy dissipative device at a fixed temperature T, biased at a voltage V with the resulting average current $\overline{i_T(V)}$, must not supply power, on average, to the external circuit. Thus the I-V curve must lie in the first and third quadrants, *i.e.*,

$$\overline{i_T(V)}\, V \geq 0, \quad \text{for } T > 0 \text{ and all } V.$$

Since the average current is assumed to be a continuous function of the applied voltage, this also implies that the average short-circuit current for a dissipative device must be zero.[3]

---

[3]Since $\overline{i_T(V)} \geq 0$ for all $V > 0$ and $\overline{i_T(V)} \leq 0$ for all $V < 0$, the average current cannot be strictly positive or negative for $V = 0$ by continuity.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

For a lossless lumped network in thermal equilibrium with a dissipative device at constant temperature, the equilibrium distribution for inductor fluxes $\boldsymbol{\phi}$ and capacitor charges $\mathbf{q}$ must have the Gibbs (or Maxwell-Boltzmann) form [10, 11, 23],

$$\rho(\boldsymbol{\phi}, \mathbf{q})^o = A \exp\left[-E(\boldsymbol{\phi}, \mathbf{q})/kT\right], \tag{2.2}$$

where $E(\boldsymbol{\phi}, \mathbf{q})$ is the sum of all inductor and capacitor stored energies and A serves to normalize the distribution.

Thermodynamic Requirement #3: Increasing Entropy During Transients

The second law of thermodynamics must be satisfied during nonequilibrium transient behavior of any circuit driven by the fluctuations of the dissipative device. The total entropy of a circuit, *i.e.*, the sum of the entropies of the lossless elements and the thermal reservoir, must be a nondecreasing function of time, with a maximum value corresponding to the equilibrium distribution [10].

These requirements are all consequences of the second law of thermodynamics. The first requirement under short-circuit conditions and the second requirement in general govern equilibrium behavior. The first requirement with nonzero d.c. voltage limits nonequilibrium steady-state behavior. The third governs transient nonequilibrium operation.

2.3   Linear Gaussian Model

The Extended Nyquist-Johnson Model

This section considers an extended version of the Nyquist-Johnson model in which the noise source $\xi(t)$ is Gaussian and the circuit model in Fig. 2-1 holds for all equilibrium

and nonequilibrium voltages. More specifically, we assume that $\xi(t)$ is unit-amplitude, stationary, zero-mean Gaussian white noise [27] and

$$i_n(t) = \sqrt{2kTG}\,\xi(t), \tag{2.3}$$

for *all* time-varying voltages v. This extends the model far beyond the thermodynamic equilibrium regime for which it was originally proposed [1, 2].

Compliance of the extended Nyquist-Johnson linear Gaussian model with the thermodynamic requirements was exhaustively addressed in [9], which describes the behavior of general *nonlinear* LC circuits driven by this model. However, as an introduction to stochastic differential equation methods and the Fokker-Planck equation used later, the tests are applied here to simple first-order RC networks.

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

For the linear Gaussian model, the average noise current is zero and is independent of the applied voltage. Thus the average electric power dissipated in the element is always nonnegative for $G \geq 0$, and of course the short-circuit average current is automatically zero. This requirement is met.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

A (possibly nonlinear) capacitor with charge q on the upper plate and constitutive relation

$$v = f(q)$$

is attached to the left side of the noise model in Fig. 2-1. The differential equation for the resulting circuit,

$$\dot{q} = -G\,f(q) - \sqrt{2kTG}\,\xi(t), \tag{2.4}$$

is of the Langevin form [16, 20]. The link between stochastic differential equations of this sort and thermodynamic variables is provided by the Fokker-Planck equation (FPE, also known as the forward Kolmogorov equation)[4] a differential equation for the probability density $\rho(q, t)$ of solutions to stochastic differential equations. For the capacitor charge random process $q(t)$ in (2.4), the FPE takes the form

$$\dot{\rho} = \frac{\partial}{\partial q}\left[Gf(q)\, \rho + kTG\, \frac{\partial \rho}{\partial q}\right] = -\frac{\partial}{\partial q}\left[J(q)\right], \qquad (2.5)$$

where $J(q)$ is called the "probability flux" [20]. Using the stored capacitor energy $E_C(q) = \int_0^q f(q')\, dq'$, the Gibbs distribution (2.2) can be immediately written:

$$\rho^o(q) = A \exp\left[-E_C(q)/kT\right]. \qquad (2.6)$$

A simple differentiation shows that this density does in fact satisfy the equilibrium condition $\dot{\rho}^o = 0$ in (2.5). Thus the second thermodynamic requirement is also met.

Note that furthermore J itself vanishes at ρ^o. (J need only be constant for ρ^o to be an equilibrium density of (2.5)). Thus the equilibrium is "detail balanced" in the language of statistical physics [20] or, equivalently, "reversible" in the language of random processes. Reversibility is an additional physical requirement for reciprocal RC circuits that does *not* hold for general RLC circuits [20, 28].

Thermodynamic Requirement #3: Increasing Entropy During Transients

The entropy S_C of the capacitor charge distribution is given by the traditional formula [11, 21]:

$$S_C \triangleq -k \int \rho \ln \rho\, dq, \qquad (2.7)$$

---

[4]A brief introduction is found in [12, pp. 650-54]; more mathematical rigor is found in [27, p. 172]; more physical intuition is found in [20, Chap. 8]; and the authors found [16, Sec. 5.2] to be generally helpful.

where k is Boltzmann's constant. (Some authors differ by an additive or multiplicative constant of no interest here.) The capacitor entropy rate is then

$$
\begin{aligned}
\dot{S}_C &= \frac{d}{dt}\left(-k\int \rho \ln \rho \, dq\right) = -k\int_{-\infty}^{+\infty} \frac{d}{dt}\left(\rho \ln \rho\right) dq \\
&= -k\int_{-\infty}^{+\infty}\left(\dot{\rho}\ln\rho + \rho\,\frac{1}{\rho}\,\dot{\rho}\right)dq = -k\int_{-\infty}^{+\infty}\dot{\rho}\ln\rho\,dq - k\int_{-\infty}^{+\infty}\dot{\rho}\,dq. \,(2.8)
\end{aligned}
$$

The final term must integrate to zero, since the total probability must remain equal to one. Before attempting to compute the first integral, we also seek an expression for the rate of change of thermal reservoir entropy. The thermodynamic identity $d\overline{E} = T\,dS$ and its time-dependent form $\frac{d\overline{E}}{dt} = T\frac{dS}{dt}$ relate the heat flow into the reservoir to its entropy. By conservation of energy, this heat flow is equal to the energy flow out of the capacitor. Thus we obtain for the time derivative of the reservoir entropy

$$
\dot{S}_R = \frac{1}{T}\frac{d}{dt}\left(-\overline{E}_C\right) = -\frac{1}{T}\frac{d}{dt}\int_{-\infty}^{+\infty}E_C(q)\,\rho\,dq = -\frac{1}{T}\int_{-\infty}^{+\infty}E_C(q)\,\dot{\rho}\,dq. \qquad (2.9)
$$

Combining the two entropy rate terms yields the total entropy rate $\dot{S}_{tot}$:

$$
\dot{S}_{tot} = \dot{S}_C + \dot{S}_R = -k\int_{-\infty}^{+\infty}\dot{\rho}\ln\rho\,dq - \frac{1}{T}\int_{-\infty}^{+\infty}E_C(q)\,\dot{\rho}\,dq, \qquad (2.10)
$$

which, using (2.5), becomes

$$
\dot{S}_{tot} = \int_{-\infty}^{+\infty}\left[-k\ln\rho - \frac{1}{T}E_C(q)\right]\frac{\partial}{\partial q}\left(G\,f\rho + kTG\frac{\partial\rho}{\partial q}\right)dq. \qquad (2.11)
$$

Integrating by parts, noting that ρ and its derivative fall off to zero very quickly at infinity so that the product term vanishes there, and recalling that $dE_C/dq = f$, we have

$$
\dot{S}_{tot} = \int_{-\infty}^{+\infty}\left[k\frac{1}{\rho}\frac{\partial\rho}{\partial q} + \frac{1}{T}f\right]\left(G\,f\rho + kTG\frac{\partial\rho}{\partial q}\right)dq
$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\rho\,GT} \left(G\,f\rho + kTG\,\frac{\partial\rho}{\partial q} \right)^2 dq \ \geq 0. \tag{2.12}$$

As we hoped, the entropy rate will always be non-negative, since it is the integral of a squared quantity.

Thus the Gaussian Nyquist-Johnson noise model for a linear resistor satisfies the equilibrium thermodynamic requirements, and the extended Nyquist-Johnson model satisfies the nonequilibrium requirements. Appendix B shows that the resistor may be time-varying; this is of interest for $1/f$ noise models based on transconductance fluctuations in the channel of a MOS device.

2.4 Nonlinear Gaussian Models

The total current through any nonlinear resistor at any fixed voltage V and temperature T can be written as the sum of an average current $g_T(V)$ and a zero-mean noise current with some power spectral density $S_{ii}(\omega; T, V)$. Note that this statement does not make any assumptions about the device itself, but merely states a fact about probability: any random signal can be represented as the sum of its mean and a zero-mean fluctuation. In many models, $e.g.$, [4, 29], the noise is white, and thus the current can be written in the form

$$i(t) = g_T(V) + h_T(V)\,\xi(t), \tag{2.13}$$

where $\xi(t)$ is unit-amplitude, stationary, zero-mean white noise. It follows from (2.13) that at each fixed V and T,

$$\bar{i} = g_T(V)$$
$$S_{ii}(\omega; T, V) = h_T^2(V), \quad \text{for all } \omega.$$
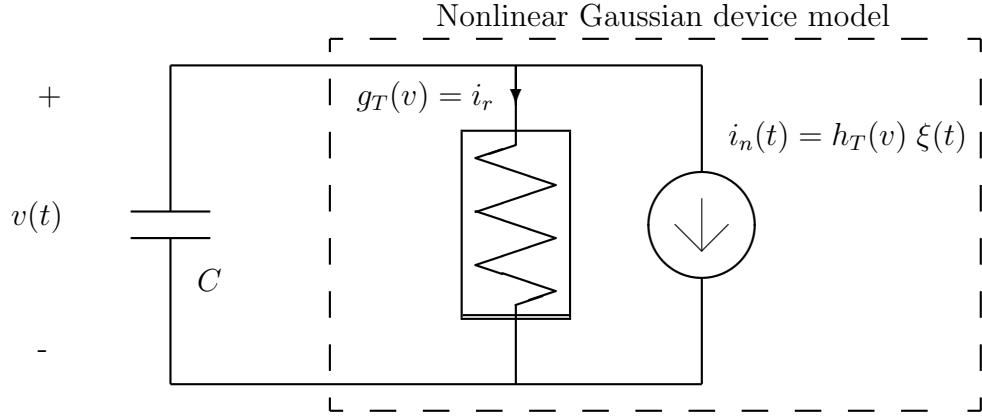
Figure 2-3: Nonlinear Gaussian device model connected to a linear capacitor. The noise current amplitude varies instantaneously with the applied voltage in this model.

This section considers the analytically simplest special class of such models where $\xi(t)$ is Gaussian and (2.13) holds for time-varying voltages at each instant, *i.e.*,

$$i = g_T(v(t)) + h_T(v(t))\, \xi(t).$$

(See Fig. 2-3.) These models are a natural extension of the linear Gaussian model in Section 2.3.

We will show a somewhat surprising result: no nonlinear device can be described by a model in this class that meets the equilibrium thermodynamic requirement (Requirement #2), regardless of the choice of $h_T(v)$. We only need a linear capacitor to illustrate the problem. This lets us focus on the voltage rather than the charge, since $\rho_v(v,t)dv = \rho_q(q,t)dq$, *i.e.*,

$$\rho_v(v,t) = \rho_q(q,t)\frac{dq}{dv} = C\rho_q(Cv,t) \tag{2.14}$$

where ρ_q is the probability density for charge and ρ_v is the probability density for voltage.

The stochastic differential equation (2.15), a nonlinear variant of the Langevin

equation, describes the dynamics of the capacitor voltage in Fig. 2-3:

$$\dot{v}(t) = -\frac{g_T(v)}{C} - \frac{h_T(v)}{C}\xi(t). \tag{2.15}$$

Certain technical problems arise with this equation because white noise of unlimited bandwidth is a mathematical fiction. These problems become especially severe in (2.15) because $h_T(v)$ can vary with v, in contrast to the usual Langevin equation. The literature focuses on two interpretations for the integral of (2.15), the Itô and the Stratonovich integrals [20, 30]. (See Appendix 3.6.) The interpretations lead to different densities ρ^I and ρ^S (corresponding to the Itô and Stratonovich interpretations) for the capacitor voltage.

The Fokker-Planck equation (FPE) for the Itô interpretation of (2.15) is:

$$\frac{\partial \rho^I}{\partial t} = \frac{\partial}{\partial v}\left\{\frac{g_T(v)}{C}\rho^I(v,t) + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h_T^2(v)}{C^2}\rho^I(v,t)\right]\right\} = -\frac{\partial}{\partial v}\left[J^I(v)\right], \tag{2.16}$$

where $J^I(v)$ is the probability flux, as in (2.5). The Stratonovich FPE for ρ^S contains one additional term:

$$\begin{aligned}
\frac{\partial \rho^S}{\partial t} &= \frac{\partial}{\partial v}\left\{\frac{g_T(v)}{C}\rho^S(v,t) - \frac{h_T(v)p^S(v,t)}{2\,C^2}\frac{\partial}{\partial v}h_T(v) + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h_T^2(v)}{C^2}\rho^S(v,t)\right]\right\} \\
&= -\frac{\partial}{\partial v}\left[J^S(v)\right]. \tag{2.17}
\end{aligned}$$

Whichever interpretation is used, the equilibrium solution for charge must fit the Gibbs form (2.2). Equivalently, using (2.14), we require

$$\rho_v^o(v) = \frac{\exp\left(-Cv^2/2kT\right)}{\sqrt{2\pi kT/C}}, \tag{2.18}$$

which happens to be Gaussian only because the capacitor is linear with energy $E =$

$\frac{1}{2}Cv^2$. Noting that

$$\frac{\partial \rho_v^o(v)}{\partial t} = 0$$

and recalling from Section 2.3 that $J^I(v)$ and $J^S(v)$ must vanish identically for RC circuits, we substitute $\rho_v^o(v)$ from (2.18) into (2.16) and arrive at the differential equation

$$\frac{\partial h_T^2(v)}{\partial v} = C \left[\frac{v}{kT} \, h_T^2(v) - 2 \, g_T(v) \right], \tag{2.19}$$

or into (2.17) to arrive at

$$\frac{\partial h_T^2(v)}{\partial v} = 2C \left[\frac{v}{kT} \, h_T^2(v) - 2 \, g_T(v) \right]. \tag{2.20}$$

Since $h_T^2(v)$ is a characteristic of the device model, it cannot depend on the value of C. The only solutions of (2.19) and (2.20) that do not vary with C are those for which the term in brackets vanishes, *i.e.*,

$$h_T^2(v) = 2kT \, \frac{g_T(v)}{v}.$$

On the left side, this implies that

$$\frac{\partial h_T^2(v)}{\partial v} \equiv 0. \tag{2.21}$$

Together, these last two equations imply that $g_T(v)/v$ is constant, *i.e.*,

$$\frac{g_T(v)}{v} = G, \quad \text{for all } v. \tag{2.22}$$

Thus, we have concluded that for both the Itô and Stratonovich interpretations for (2.15), in order to have the correct equilibrium distribution, the resistor *must* be a linear resistor with

$$i = Gv,$$

and the resulting noise amplitude,

$$h_T^2(v) = 2kTG,$$

is precisely that from the the traditional Nyquist-Johnson model for the linear case.

This calculation has shown that no resistor with a nonlinear constitutive relation $\bar{i} = g_T(v)$ has a Gaussian white noise-current model[5] of the form shown in Fig. 2-3, even within the special domain of thermal equilibrium. This calculation also gave an independent derivation of the Gaussian Nyquist-Johnson model for a linear resistor at thermal equilibrium.

Nyquist's derivation used two resistors connected to a transmission line (a distributed LC circuit) and required the equipartition theorem to be satisfied by the energy in the modes of the transmission line. Our derivation uses a simpler circuit, consisting of only one resistor and one capacitor. However, the Gibbs distribution is a more stringent requirement than the equipartition theorem, since other non-thermodynamic distributions satisfy the equipartition theorem.

2.5 Shot-Noise Models

2.5.1 Poisson Models for Shot Noise

The shot-noise model for a current of electrons or holes describes the arrival of each charged particle as a Dirac delta function of current

$$\pm e\, \delta(t - t_n),$$

where t_n is the n-th arrival time, $e > 0$ is the magnitude of the electron charge, and the sign is chosen positive for a hole and negative for an electron. The arrival times

---

[5]at least in the Itô and Stratonovich interpretations of (2.19)

are randomly distributed. If we further require that the distribution of the arrival times be *memoryless*, that is,

$$\Pr(t_n - t_{n-1} > t + h \mid t_n - t_{n-1} > t) = \Pr(t_n - t_{n-1} > h),$$

we obtain the *Poisson point process* (PPP), which is a Markov process [13]. A *homogeneous* Poisson point process is stationary, *i.e.*, the *average arrival rate* λ is constant. In a shot-noise model, this would mean that the expected number of arrivals in any time interval of length Δt is $\lambda \Delta t$, and the average current is $\pm e \lambda$.

However, λ need not be constant, in which case we obtain an *inhomogeneous* Poisson point process, which is not stationary. The expected number of arrivals in any interval $[t, t + \Delta t]$ is

$$\int_t^{t+\Delta t} \lambda(\tau) \, d\tau.$$

If we connect our shot-noise source to a capacitor, the charge on the capacitor will be given by the familiar Poisson counting process (PCP), the integral of the PPP with respect to time, as seen in Fig. 2-4.

For the following derivations, it will be useful to note that one can reparameterize the time axis such that an inhomogeneous PCP can be expressed as a homogeneous PCP on a non-uniform time axis. Let $N(t)$ be a PCP with rate 1. Then to generate an inhomogeneous PCP N_{inhom} with the rate $\lambda(t)$, let

$$N_{inhom}(t) = N\left(\int_0^t \lambda(\tau)d\tau\right). \tag{2.23}$$

The random process N_{inhom} is still Markovian, with independent increments. [13, 31]

2.5.2 Poisson Device Models

A two-terminal Poisson device model (*i.e.*, a shot-noise model) consists simply of two independent forward and reverse current random processes. (See Fig. 2-5.)
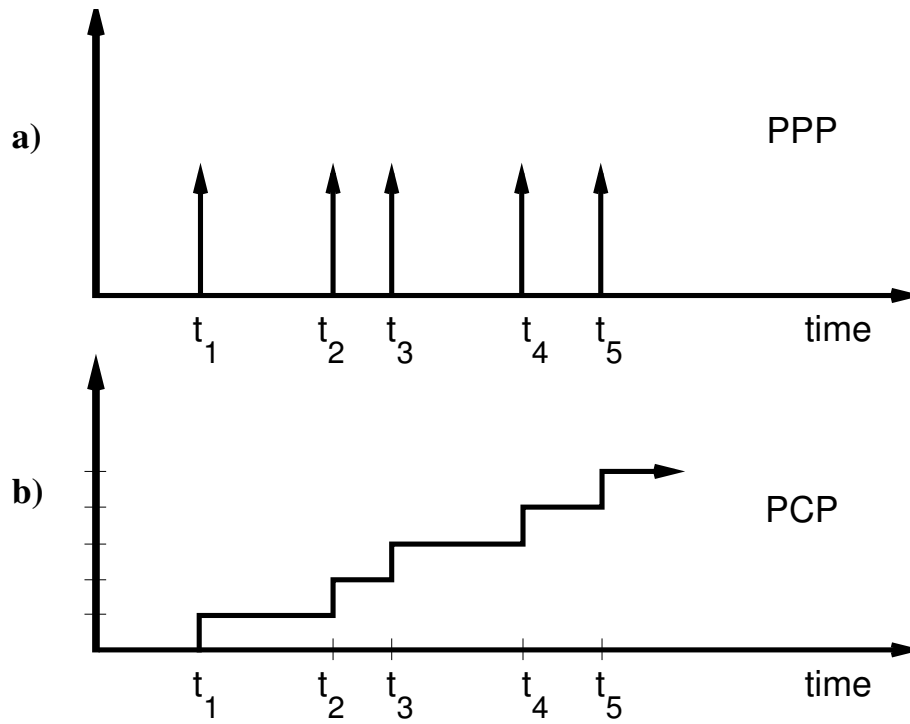
Figure 2-4: Possible behavior of (a) the Poisson point process and (b) the corresponding counting process.
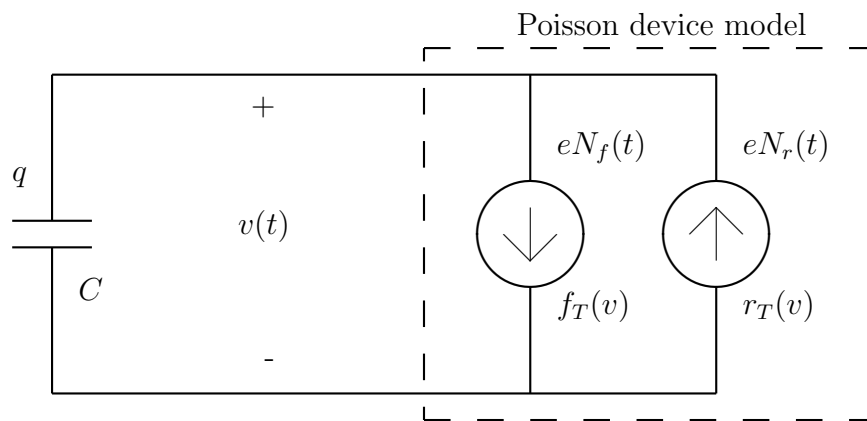
Figure 2-5: Poisson device model connected to a capacitor. The forward current source, $eN_f(t)$, has a voltage-dependent average arrival rate $f_T(v)$; similarly for the reverse current.

Each current is a Poisson counting process with a rate λ that is a function of the instantaneous applied voltage v and the temperature T, $i.e.$,

$$i(t) = \frac{d}{dt} \left\{ eN_f \left(\int_0^t f_T(v(\tau))d\tau \right) - eN_r \left(\int_0^t r_T(v(\tau))d\tau \right) \right\}, \qquad (2.24)$$

where N_f and N_r are the independent homogeneous forward and reverse counting processes, and $f_T(v)$ and $r_T(v)$ the forward and reverse rates, $i.e.$,

$$
\begin{aligned}
\lambda_f = f_T(v) &> 0, \qquad \text{for all } v \text{ and } T > 0, \\
\lambda_r = r_T(v) &> 0, \qquad \text{for all } v \text{ and } T > 0.
\end{aligned}
\qquad (2.25)
$$

Note that the Poisson device model incorporates both the deterministic constitutive relation for the device as well as the stochastic noise behavior: the average current is

$$\overline{i(t)} = e\left[f_T(v(t)) - r_T(v(t)) \right],$$

and the constitutive relation for the device ($i.e.$, the $v - i$ curve) is

$$\overline{i(v)} = e\left[f_T(v) - r_T(v) \right]. \qquad (2.26)$$

Under d.c. bias conditions with constant V, the current random process $i(t)$ becomes stationary and hence has a power spectral density. The spectrum is white, apart from the d.c component [12], with magnitude

$$S_{ii}(\omega; T, V) = e^2 \left[f_T(V) + r_T(V) \right], \qquad \text{for } \omega \neq 0. \qquad (2.27)$$

The analytical simplicity of this model comes from the three *very strong* assumptions that 1) the electron arrival is instantaneous and can therefore be modeled as a δ-function, 2) the two random processes are mutually independent and memoryless,

and 3) the expected arrival rate changes instantaneously with v.

For some devices, this model is reasonably accurate over a wide enough range of d.c. bias voltages to include substantially nonlinear portions of the $v-i$ curve. The pn junction and the MOSFET in the subthreshold regime are two interesting examples. One would expect this model also applies to other devices under nonequilibrium bias conditions, provided a) the lattice remains at a uniform constant temperature during such operation, and b) the carrier population remains locally in thermal equilibrium with the lattice, *i.e.*, retains approximately the Gibbs distribution at a constant temperature T, throughout the device during such operation.

Since the noise statistics are determined by the *sum* of the average currents (2.27) while the constitutive relation is determined by the *difference* (2.26), the development so far does not imply any unique relation between the constitutive relation and the noise. We will show that with the thermodynamic requirements, the constitutive relation and the temperature *uniquely specify* the current noise at each operating voltage V.

Example: Subthreshold MOSFET

The subthreshold p-channel MOSFET with fixed gate-to-source voltage V_{gs} is a two-terminal device that is well-described by a Poisson model. The derivation of this model and a comparison with experimental results is given in [32]. There are only two currents, i_f and i_r, and both are hole diffusion currents in the n-region shown in Fig. 2-6. The separation of the total currents into forward and reverse currents in this model is done as follows: given the hole concentration at both ends, the current from each end is calculated as the diffusion that would occur if the concentration at the far end were zero. In this model,

$$\begin{aligned}
\overline{i_f} &= ef_T(v) = I_{sat}(V_{gs}) \\
\overline{i_r} &= er_T(v) = I_{sat}(V_{gs}) \exp(-ev/kT),
\end{aligned}$$
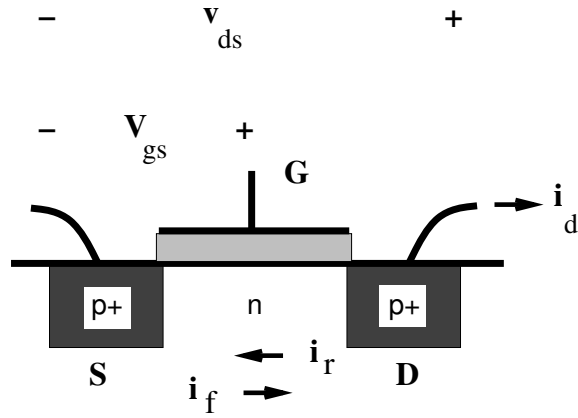
Figure 2-6: MOSFET cross-section.

where $v = v_{ds}$ is the drain-to-source voltage, so that

$$\overline{i_d} = \overline{i_f} - \overline{i_r} = I_{sat}(V_{gs})[1 - \exp(-ev/kT)],$$

and the shot-noise amplitude is given by the sum

$$S_{ii}(\omega; T, V) = e(\overline{i_f} + \overline{i_r}), \quad \text{for } \omega \neq 0.$$

Example: PN Junction

To develop a shot-noise model for the *pn* junction in Fig. 2-7, we need expressions for the forward and reverse currents. The dominant currents are the electron and hole diffusion currents.

Diffusion currents result from the differences in carrier concentrations on opposite sides of the junction. At the edge of the space charge region on the p side, the electron concentration is $n_{po} \exp(eV/kT)$, but deep in the bulk p region, the electron
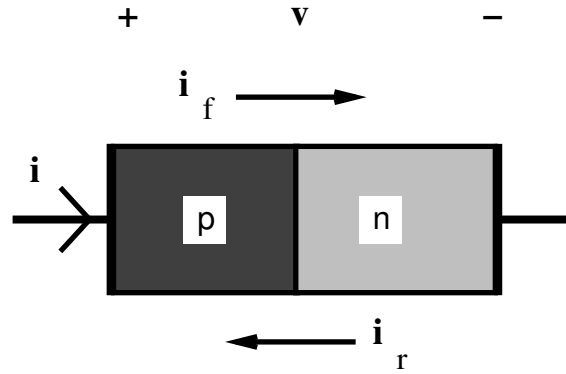
Figure 2-7: A *pn* junction.

concentration is n_{po}. The electron diffusion current, therefore, is proportional to

$$n_{po} \left[\exp(eV/kT) - 1 \right].$$

The hole diffusion concentration is proportional to a similar factor,

$$p_{no} \left[\exp(eV/kT) - 1 \right].$$

Although electrons and holes diffuse in opposite directions, the currents are in the same direction, yielding a net average current

$$\bar{i} = I_S \left[\exp(ev/kT) - 1 \right],$$

where the saturation current I_S incorporates all the constants, such as the bulk carrier concentrations and diffusion coefficients.

Dividing the current into forward and reverse currents in this model is not as clearly justified as it was in the MOSFET case. Nevertheless, following the philosophy of the alternate derivation of noise for the linear resistor in [32], we take the concentration near the electrode, in this case the electron concentration deep in the

bulk p region, to determine the concentration for the reverse current of electrons. Correspondingly, we get a reverse current of holes from their concentration deep in the n region. This results in a total reverse current (of holes and electrons)

$$\overline{i_r} = I_S = er_T(v)$$

and a forward current

$$\overline{i_f} = I_S \exp(ev/kT) = ef_T(v).$$

Shot noise is generated by both currents, and for fixed V, the power spectral density is

$$S_{ii}(\omega; T, V) = e(\overline{i_f} + \overline{i_r}), \quad \text{for } \omega \neq 0.$$

More physical detail can be found in most semiconductor device textbooks. For more details on the noise model, the reader is referred to [29].

2.5.3   Thermodynamic Tests on Poisson Models

Thermodynamic Requirement #1: No Isothermal Conversion of Heat to Work

The requirement is that

$$Ve\left[f_T(V) - r_T(V)\right] \geq 0, \quad \text{for } T > 0 \text{ and all } V. \tag{2.28}$$

It is satisfied for both the subthreshold MOSFET and the pn junction shot-noise models.

Thermodynamic Requirement #2: Gibbs Distribution at Equilibrium

For this second test, we consider our noisy device in a circuit with a single linear capacitor, as in Fig. 2-5. The equilibrium distribution of charge on this capacitor

must have the Gibbs form.

Integrating the circuit differential equation $\frac{dq}{dt} = -i$ and using the device current from equation (2.24), we find

$$q(t) = -e \left\{ N_f \left(\int_0^t f_T(q(\tau)/C)d\tau \right) - N_r \left(\int_0^t r_T(q(\tau)/C)d\tau \right) \right\}, \qquad (2.29)$$

and we choose the initial condition $q(0) = 0$. (For mnemonics, recall that in defining the device model, f_T was used for "forward" current and r_T for "reverse" current, with respect to the sign conventions for the device. But in the circuit, it is better to think of f_T as standing for "falling" charge and r_T for "rising" charge on the capacitor.) Note that the rates $f_T(q(t)/C)$ and $r_T(q(t)/C)$ are discontinuous functions of time, since the capacitor can only have integer numbers of electrons on its plates. This raises a question about interpreting the transition rates correctly. Should we use the charge value before the jump, the value afterwards, or the average?

It turns out that using the charge values before the jump mishandles the discontinuities in $f_T(v(t))$ and $r_T(v(t))$: in the subthreshold MOSFET and pn junction examples, it results in an equilibrium charge distribution that is not Gibbsian and has a mean value of $-\frac{1}{2}e$, contrary to the requirement. For this reason we let the transition rate be governed by the average of the capacitor voltages before and after the jump. Using simplified notation for the transition rates

$$r_n \triangleq r_T \left(\frac{(n+1/2)e}{C} \right) \qquad (2.30)$$

$$f_n \triangleq f_T \left(\frac{(n-1/2)e}{C} \right), \qquad (2.31)$$

and for the conditional probabilities

$$p(n,t \,|\, m,s) \triangleq \Pr\{q(t) = ne \,|\, q(s) = me\},$$
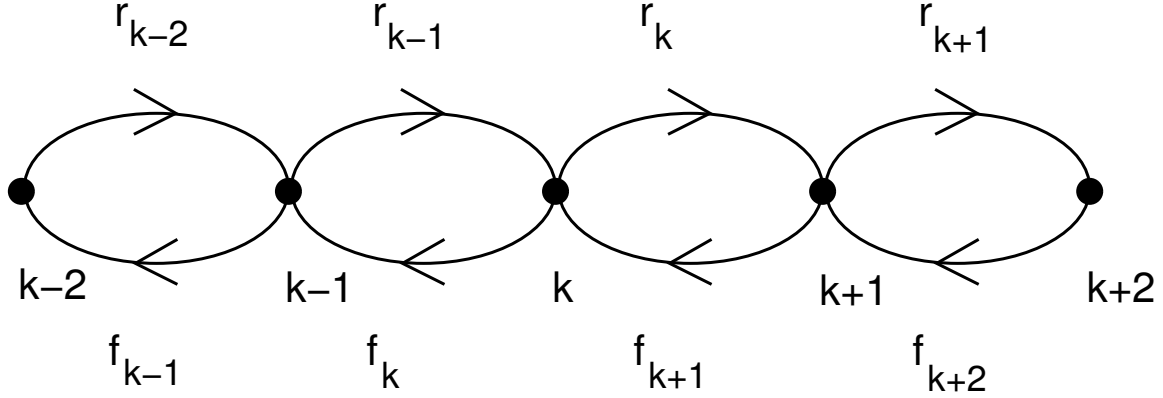
Figure 2-8: Section of the capacitor charge Markov Chain (2.32). Node k represents the state with charge $+ke$ on the upper capacitor plate.

one arrives at the forward evolution equation for the probability distribution (*i.e.*, the Master Equation [20])

$$\frac{d}{dt}p(n,t) \;\; = \;\; r_{n-1}\,p(n-1,t) + f_{n+1}\,p(n+1,t) - [r_n + f_n]\,p(n,t). \quad (2.32)$$

For more detail, see [33]. These transition probabilities describe the infinite Markov chain in Fig. 2-8.

The equilibrium distribution p_n^o satisfies (2.32) with the left hand side set to zero. Again requiring detailed balance, the total flow between adjacent nodes must vanish, *i.e.*,

$$r_n\,p_n^o = f_{n+1}\,p_{n+1}^o, \quad \text{for each } n, \quad (2.33)$$

or

$$\frac{p_{n+1}^o}{p_n^o} = \frac{r_n}{f_{n+1}}, \quad \text{for each } n. \quad (2.34)$$

The equilibrium solution can quickly be found in closed form (except, perhaps, for

normalization):

$$\frac{p_n^o}{p_0^o} = \begin{cases} \displaystyle\prod_{j=0}^{n-1} \frac{r_j}{f_{j+1}}, & n > 0 \\[2ex] \displaystyle\prod_{j=0}^{n+1} \frac{f_j}{r_{j-1}}, & n < 0. \end{cases} \tag{2.35}$$

We may now test this distribution for consistency with the Gibbs form. Gibbs statistics for our circuit require that the ratio of probabilities of neighboring states satisfy

$$\begin{aligned} \frac{p_{n+1}^o}{p_n^o} &= \frac{\exp\left[-\left(\frac{(n+1)^2 e^2}{2CkT}\right)\right]}{\exp\left[-\left(\frac{n^2 e^2}{2CkT}\right)\right]} = \exp\left[-\left(\frac{(2n+1)e^2}{2CkT}\right)\right] \\ &= \exp\left[-\left(\frac{(2n+1)e}{2Cv_T}\right)\right] = \exp\left[-\left(\frac{ne}{Cv_T}\right)\right]\exp\left[-\left(\frac{e}{2Cv_T}\right)\right]. \end{aligned} \tag{2.36}$$

The Markov chain for the circuit also gives an expression (2.34) for the ratios of the neighboring equilibrium probabilities:

$$\frac{p_{n+1}^o}{p_n^o} = \frac{r_n}{f_{n+1}} = \frac{r_T((n+1/2)e/C)}{f_T((n+1/2)e/C)} = \frac{r_T(v_n + e/2C)}{f_T(v_n + e/2C)}, \tag{2.37}$$

where v_n is the voltage on the capacitor when the upper plate stores n positive charges. Equation (2.37) agrees with the thermodynamic requirement (2.36) for all capacitors if and only if

$$\frac{r_T(v)}{f_T(v)} = \exp\left(-v/v_T\right), \quad \text{for all } v. \tag{2.38}$$

The probability ratio (2.37) from the Markov chain becomes

$$\begin{aligned} \frac{p_{n+1}^o}{p_n^o} &= \frac{r_T(v_n + e/2C)}{f_T(v_n + e/2C)} = \exp\left(-\frac{v_n + e/2C}{v_T}\right) \\ &= \exp\left(-ne/Cv_T\right)\exp\left(-e/2Cv_T\right), \end{aligned}$$

which agrees precisely with (2.36).

Thus the constraint (2.38) is both necessary and sufficient to guarantee that ev-

ery shot-noise model leads to a Gibbs equilibrium distribution of charge on a linear capacitor, as required by thermodynamics. We have also shown that this conclusion continues to hold even when the capacitor is nonlinear [33]. Both the *pn* junction and the subthreshold MOSFET shot noise models satisfy (2.38).

Furthermore, given the positivity restrictions (2.25), it is easy to show that the constraint (2.38) guarantees Thermodynamic Requirement #1 is also satisfied.

Thermodynamic Requirement #3: Increasing Entropy During Transients

Ref. [34] has a proof based on information theory and the "relative entropy." A new proof is presented below.

The capacitor entropy is defined as in Eq. (2.7), but for a discrete state space,

$$S_C \overset{\Delta}{=} -k \sum_{n=-\infty}^{+\infty} p(n,t) \, \ln p(n,t) \tag{2.39}$$

The capacitor entropy rate is

$$
\begin{aligned}
\dot{S}_C &= -k \sum_{n=-\infty}^{+\infty} \left(\dot{p}(n,t) \, \ln p(n,t) + p(n,t) \, \frac{d}{dt} \ln p(n,t) \right) \\
&= -k \sum_{n=-\infty}^{+\infty} \left(\dot{p}(n,t) \, \ln p(n,t) + p(n,t) \, \frac{1}{p(n,t)} \, \dot{p}(n,t) \right) \\
&= -k \sum_{n=-\infty}^{+\infty} \dot{p}(n,t) \, \ln p(n,t), \tag{2.40}
\end{aligned}
$$

because the second term sums to zero by conservation of probability.

Using the First Law argument as in Section 2.3, the resistor entropy rate is

$$\dot{S}_R = \frac{1}{T} \frac{d}{dt} \left(-\overline{E}_C \right) = -\frac{1}{T} \frac{d}{dt} \sum_{n=-\infty}^{+\infty} \frac{(ne)^2}{2C} \, p(n,t) = -\frac{1}{T} \sum_{n=-\infty}^{+\infty} \frac{(ne)^2}{2C} \, \dot{p}(n,t). \tag{2.41}$$

Combining the two entropy rate terms yields the total entropy rate $\dot{S}_{tot}$:

$$
\begin{aligned}
\dot{S}_{tot} &= \dot{S}_C + \dot{S}_R = -k \sum_{n=-\infty}^{+\infty} \dot{p}(n,t) \ln p(n,t) - \frac{1}{T} \sum_{n=-\infty}^{+\infty} \frac{(ne)^2}{2C} \dot{p}(n,t) \\
&= -\sum_{n=-\infty}^{+\infty} \dot{p}(n,t) \left(k \ln p(n,t) + \frac{e^2}{2CT} n^2 \right)
\end{aligned}
\tag{2.42}
$$

Substituting in the Master Equation (2.32) for $\dot{p}(n,t)$,

$$
\begin{aligned}
\dot{S}_{tot} &= -\sum_{n=-\infty}^{+\infty} \Big[r_{n-1}\, p(n-1,t) + f_{n+1}\, p(n+1,t) - [r_n + f_n]\, p(n,t) \Big] \\
&\qquad\qquad \times \left(k \ln p(n,t) + \frac{e^2}{2CT} n^2 \right) \\
&= \sum_{n=-\infty}^{+\infty} \Big[r_n\, p(n,t) - f_{n+1}\, p(n+1,t) \Big] \left(k \ln p(n,t) + \frac{e^2}{2CT} n^2 \right) \\
&\quad + \sum_{n=-\infty}^{+\infty} \Big[-r_{n-1}\, p(n-1,t) + f_n\, p(n,t) \Big] \left(k \ln p(n,t) + \frac{e^2}{2CT} n^2 \right)
\end{aligned}
$$

The terms in square brackets on the last two lines are offset by one, so we can reindex and then recombine.

$$
\begin{aligned}
\dot{S}_{tot} &= \sum_{n=-\infty}^{+\infty} \Big[r_n\, p(n,t) - f_{n+1}\, p(n+1,t) \Big] \\
&\qquad \times \left(k \ln p(n,t) + \frac{e^2}{2CT} n^2 - k \ln p(n+1,t) - \frac{e^2}{2CT}(n+1)^2 \right) \\
&= \sum_{n=-\infty}^{+\infty} \Big[r_n\, p(n,t) - f_{n+1}\, p(n+1,t) \Big] \left(k \ln \frac{p(n,t)}{p(n+1,t)} - \frac{e^2}{2CT}(2n+1) \right)
\end{aligned}
$$

Recall the thermodynamic constraint (2.38) and the definitions of f_n and r_n in Eqs. (2.31) and (2.30). Factoring suggestively,

$$
\begin{aligned}
\dot{S}_{tot} &= \sum_{n=-\infty}^{+\infty} k\, r_n\, p(n+1,t) \left[\frac{p(n,t)}{p(n+1,t)} - \exp\left(\frac{(n+1/2)e}{C v_T} \right) \right] \\
&\qquad \times \left(\ln \frac{p(n,t)}{p(n+1,t)} - \frac{e^2}{2CkT}(2n+1) \right).
\end{aligned}
\tag{2.43}
$$

The rate r_n is positive by assumption in Eq. (2.25); the probabilities $p(n,t)$ are by definition non-negative; and Boltzmann's constant k is positive. Note that the terms in the large parentheses are the logarithms of those in the square brackets, since

$$v_T = \frac{kT}{e} \Rightarrow \frac{(n+1/2)e}{Cv_T} = \frac{e^2}{2CkT}(2n+1).$$

Because the logarithm is monotonic,

$$(a - b)\ (\ln a - \ln b) > 0,$$

unless $a = b$. Therefore, the total entropy is the sum of infinitely many non-negative terms,

$$\dot{S}_{tot} \geq 0, \tag{2.44}$$

with equality if and only if

$$\frac{p(n,t)}{p(n+1,t)} = \exp\left(\frac{(n+1/2)e}{Cv_T}\right),$$

which is precisely the Gibbs relationship (2.36).

Summary

In summary, the shot-noise model satisfies all the thermodynamic requirements presented here if and only if the forward and reverse rates are related by (2.38), which applies to both time-varying and d.c. voltages. After developing this model, we discovered a distinct but related treatment in [21]. His derivation is based on a complicated "kinetic potential" argument. It uses an approximation [21, eq. (3.3.43)], not used or needed here and handles the discontinuities in $v(t)$ differently. In addition, we have explicitly verified that the Poisson model satisfies the increasing entropy requirement.

For d.c. voltages V, the constraint (2.38) leads to a prediction of a unique current

noise amplitude at each operating point. If we define

$$\bar{i} = g(V) = e\left[f_T(V) - r_T(V)\right] = e\left[\exp\left(V/v_T\right) - 1\right]r_T(V), \qquad (2.45)$$

then for all $\omega \neq 0$,

$$
\begin{aligned}
S_{ii}(\omega; T, V) &= e^2\left[f_T(V) + r_T(V)\right] = e^2\left[\exp\left(V/v_T\right) + 1\right]r_T(V) \\[2mm]
&= \frac{e\left[\exp\left(V/v_T\right) + 1\right]}{\left[\exp\left(V/v_T\right) - 1\right]}\,g(V) = \frac{e\,g(V)}{\tanh\left(V/2v_T\right)}, \qquad (2.46)
\end{aligned}
$$

at each d.c. voltage V.

2.6 Comparison Between Shot-Noise and Extended Nyquist-Johnson Models

The two thermodynamically acceptable models, Nyquist-Johnson and shot, are fundamentally distinct since the former is Gaussian and the latter is not. But their power spectra are both white and can be compared. For a device with average current given by $g_T(V)$ at a fixed operating voltage V and temperature T, we compare the Poisson model power spectral density (2.46) with the value S_{ii}^{NJ} that the Nyquist-Johnson model would predict if one applied it to the linearized conductance $g_T'(V)$,

$$S_{ii}^{NJ} = 2kTg_T'(V). \qquad (2.47)$$

It is reassuring to note that the Poisson (2.46) and Nyquist-Johnson (2.47) power spectral densities agree in the short-circuit case. This can be seen by expanding (2.46) about $V = 0$ using l'Hôpital's rule. But they do not agree elsewhere in general. *Note that there is no reason to believe (2.47) gives a correct prediction for any nonlinear*

device with $V \neq 0$, despite its occasional use in the literature.

To push the comparison further, we apply both models to a linear conductor G. The Poisson model (2.46) reduces to

$$S_{ii}^P = \frac{e\,G\,V}{\tanh(V/2v_T)}, \tag{2.48}$$

while the Nyquist-Johnson model, of course, gives

$$S_{ii}^{NJ} = 2kTG. \tag{2.49}$$

It is interesting that two noise models with *different* power spectral densities are *both* thermodynamically acceptable. The Poisson model predicts a larger current noise than the Nyquist-Johnson model at each nonzero bias point, since

$$\frac{S_{ii}^P}{S_{ii}^{NJ}} = \frac{V/2v_T}{\tanh(V/2v_T)} > 1, \quad \text{for all } V \neq 0.$$

The shot-noise model is "noisier" than the extended Nyquist-Johnson model for $V \neq 0$. This is a direct result of the finite size of the electron charge. To see this, consider a hypothetical family of linear conductors, all having the same conductance G and temperature T, but in which the charge quantum e comes in various sizes. (These are rare or nonexistent in electronics, but the Ca^{++} channel in nerve membrane is one example of a non-unity charge quantum.) The limiting behavior of (2.48) is

$$S_{ii}^P \to \lim_{e \to 0} \frac{eGV}{\tanh(\frac{eV}{2kT})} = 2kTG = S_{ii}^{NJ}, \tag{2.50}$$

i.e., the shot noise magnitude converges to the extended Nyquist-Johnson noise amplitude as the charge quantum vanishes.

A closer analysis shows that for any nonzero V, S_{ii}^P grows monotonically with e as e increases from zero: the larger the charge quantum, the larger the noise.

This closer analysis also exposes an odd fact about the shot-noise model for linear as well as nonlinear devices: at any fixed V, as $e \to 0$, both the forward and reverse rates grow as $1/e^2$ rather than the more intuitive $1/e$ one might expect. Thus the forward and reverse currents both become infinite in the "small charge quantum" limit of the Poisson model, while their difference remains finite. This limiting behavior is necessary so that the noise power remains nonvanishing, as can be seen from the first expression on the right-hand side of (2.46):

$$S_{ii}(\omega; T, V) = e^2 \left[f_T(V) + r_T(V) \right], \quad \text{for } \omega \neq 0,$$

where we have constrained f_T and r_T to be positive. The *net* current in (2.45) also remains finite, because

$$\exp\left(V/v_T\right) \approx 1 + \frac{eV}{kT}$$

for small e, so that

$$e \left[\exp\left(V/v_T\right) - 1 \right] r_T(V)$$

remains constant even though $r_T(V)$ grows as $1/e^2$.

The table on the following page summarizes the hypotheses and results of the two approaches.

Model	**Shot-noise**	**Extended Nyquist-Johnson**
State space	discrete	continuous
Stochastic process statistics	Poisson	Gaussian
Equilibrium condition (detailed balance)	equal forward and reverse flows in Master Equation (2.33)	probability flux J (2.5) vanishes in Fokker-Planck equation
Power spectral density	$S_{ii}^P = \frac{eg(V)}{\tanh(V/2v_T)}$ (2.46)	$S_{ii}^{NJ} = 2kTG$ (2.49)
Gibbs distribution requirement	forward and reverse rates exponentially related (2.38)	resistor must be linear (2.22)

Chapter 3

Forward Evolution Equations

In this chapter, we will examine more closely the mathematical tools that were used in the previous chapter. Most of the mathematics here is not new, and it could be argued that this material belongs in an appendix, if at all. However, a proper understanding of the Fokker-Planck equation is critical to this thesis, and the relations between some results yield valuable insights.

This chapter will start in Section 3.1 with a simple derivation of the Fokker-Planck equation from the Master Equation, which is intuitively much easier to understand. A key step in the derivation is the truncation of a series after two terms; we will also see that extending this series (the so-called Kramers-Moyal expansion) will not rescue the nonlinear Gaussian model.

Section 3.2 will present a simple Fokker-Planck equation, derived from the corresponding stochastic differential equation. Section 3.3 will discuss interpretations of the stochastic integral, which will be necessary in more complicated systems, where the white noise is multiplied by a function of the state. The interpretations include the standard Itô and Stratonovich versions.

Section 3.4 will derive the Fokker-Planck equation from Poisson counters. The section starts with a description of stochastic processes driven by Poisson counters. Gaussian white noise is obtained as a limit of a random walk, with decreasing step

size but increasing frequency. The stochastic differential equations do not have a unique interpretation because of the violent nature of the jump in the state of the system at the instant the Poisson counter increments. The main focus will be on the Itô interpretation, but the FPE for other interpretations of the stochastic differential equations will also be given. Note that it is generally believed [20, 35] that there is a unique Fokker-Planck equation corresponding to physical reality; it is only the stochastic differential equation that suffers from ambiguity of interpretation.

A natural extension of this approach to Brownian motion starting with Poisson counters is a stochastic differential equation driven by both Gaussian white noise and a Poisson counter. We will present in Section 3.5 the forward equation for the probability distribution for this case, in preparation for the next chapter, where we consider linear resistors and shot-noise models in the same circuit.

The main interest in non-Ito stochastic integrals comes from their use in the nonlinear Gaussian model. Section 3.6 considers the different interpretations for the stochastic differential equation in Chapter 2. Although one interpretation of the stochastic integral ($\zeta = 1$, which we will define) allows a nonlinear Gaussian model, this introduces "spurious drift" [20]. Further, the same model, used to describe a nonlinear device connected to an inductor rather than a capacitor as in the last chapter, again shows us that the resistor must be linear.

The last section of this chapter will show under what conditions the Poisson model will converge to a Gaussian. This analysis is motivated by the analysis of Section 2.6, where the Poisson and Gaussian had the same power spectral density for $e \to 0$, and further by the fact that the FPE was derived from Poisson counters.

3.1   The Kramers-Moyal Expansion

This section presents a heuristic derivation of the Fokker-Planck equation as the limit of a Master Equation, under the assumption that only "small" jumps occur

as the system evolves. The meaning of "small" will be made more precise later. If this assumption is not valid, then one can continue the so-called Kramers-Moyal expansion and include further terms. In [20], van Kampen says that these terms are never exactly zero for physical systems; however, in most applications we are aware of, these terms are negligible. This section concludes by showing that, even if they were not negligible, the forward equation including these terms for Gaussian models for nonlinear resistors still would not admit the correct equilibrium density.

3.1.1 FPE as the Limit of the Master Equation

This section is a recapitulation of the relevant material in [20, p. 198] and [36].

The Master Equation for the time evolution of the probability distribution is

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho(t,x) &= \int_{-\infty}^{+\infty} W(y,x)\rho(t,y) - \rho(t,x)W(x,y) \; dy \\
&= \int_{-\infty}^{+\infty} [W(x-r,x)\rho(t,x-r) - W(x,x+r)\rho(t,x)] \; dr,
\end{aligned}
$$

where $W(y,x)$ is the transition rate from y to x. The first term in the summation is the amount of probability flowing into the point x from all other points, and the second term is the probability leaving x. The second line is simply a change of variables, where $r = x - y$, because the next assumption is that $W(x-r,x)$ is smooth in x and short-range in r, that is, transitions are overwhelmingly to nearby states. Under the further assumption that $\rho(t,x)$ is also slowly-varying in x, we can perform the following Taylor expansion on the first term:

$$
W(x-r,x)\rho(t,x-r) \tag{3.1}
$$
$$
\begin{aligned}
&= W(x,x+r)\rho(t,x) + \frac{\partial}{\partial x}\left[W(x,x+r)\rho(t,x)\right](-r) \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[W(x,x+r)\rho(t,x)\right](-r)^2 \quad + \quad o(r^2), \tag{3.2}
\end{aligned}
$$

where $o(r^2)$ denotes a term with the property that $o(r^2)/r^2 \to 0$ as $r \to 0$. Therefore,

$$
\frac{\partial}{\partial t}\rho(t,x) = \int_{-\infty}^{+\infty} \left\{ W(x,x+r)\rho(t,x) + \frac{\partial}{\partial x}\left[W(x,x+r)\rho(t,x)\right](-r) \right.
$$
$$
\left. + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[W(x,x+r)\rho(t,x)\right](-r)^2 + o(r^2) \right\} dr
$$
$$
- \int_{-\infty}^{+\infty} W(x,x+r)\rho(t,x)\ dr.
$$

The first and fourth terms cancel, and we can exchange the order of integration (by r) and differentiation (by x).

$$
\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\underbrace{\left[\int_{-\infty}^{+\infty} r\,W(x,x+r)\ dr\right]}_{a_1(x)}\rho(t,x)
$$
$$
+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\underbrace{\left[\int_{-\infty}^{+\infty} r^2\,W(x,x+r)\ dr\right]}_{a_2(x)}\rho(t,x) + o(r^2). \qquad (3.3)
$$

The "jump moments" [20]

$$
a_n(x) \triangleq \int_{-\infty}^{+\infty} r^n\,W(x,x+r)\ dr, \quad n=1,2, \qquad (3.4)
$$

are related to the drift and diffusion coefficients of the FPE. Although van Kampen asserts that a_n for $n > 2$ are never exactly zero for physical systems [20, footnote **) on p. 199], they are generally neglected. If we can justify neglecting higher-order terms, this last equation has the same form as the Fokker-Planck equation:

$$
\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\left[a_1(x)\rho(t,x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[a_2(x)\rho(t,x)\right].
$$

In this way, we have shown that the FPE arises as a natural limit of the Master Equation. However, this derivation does not connect the FPE back to the stochastic differential equation. Of course, the Master Equation similarly lacks a connection to

a stochastic differential equation; in general, one must have some other procedure for determining the transition probabilities $W(y, x)$. We will show the connection for a simple system in Section 3.2 and then more generally in Section 3.4.

3.1.2 Kramers-Moyal Forward Equation

Suppose now that the transition probability $W(x, x + r)$ is not short-range in r. For nonlinear systems, one might find that the first and second order jump moments are not sufficient to describe the system. In that case, one builds the so-called Kramers-Moyal expansion. If the Taylor expansion of (3.2) is continued,

$$
\begin{aligned}
W(x - r', x)\rho(t, x - r') &= W(x, x + r)\rho(t, x) \\
&+ \frac{\partial}{\partial x}\left[W(x, x + r)\rho(t, x)\right](-r) \\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[W(x, x + r)\rho(t, x)\right](-r)^2 \\
&+ \frac{1}{3!}\frac{\partial^3}{\partial x^3}\left[W(x, x + r)\rho(t, x)\right](-r)^3 \\
&+ \quad \ldots \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial x^n}\left[W(x, x + r)\rho(t, x)\right](-r)^n,
\end{aligned}
$$

the corresponding probability evolution equation is

$$
\frac{\partial}{\partial t}\rho(t, x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}\left[a_n(x)\rho(t, x)\right], \tag{3.5}
$$

where the jump moments a_n are given by Eq. 3.4 for higher values of n.
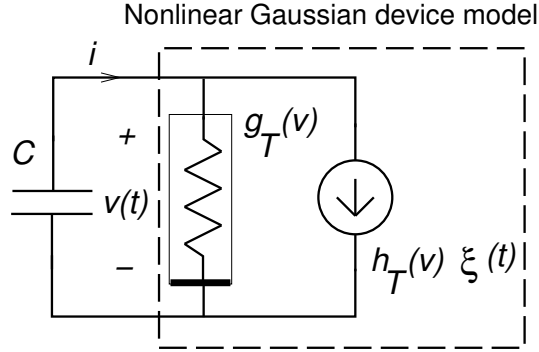
Figure 3-1: Nonlinear Gaussian device model connected to a linear capacitor

3.1.3   Nonlinear Gaussian Model

Now, let us consider the expanded Kramers-Moyal evolution equation applied to the Gaussian model for a nonlinear resistor. Specifically, will the Gibbs distribution be an equilibrium ($\dot{\rho} = 0$) for Eq. (3.5)?

Weiss and Mathis claim [37] that adding more terms in the expansion to describe a nonlinear system will require "additional information" not determined by network theory; they offer that these terms might be determined by experiment. This subsection will explore the possibility that the thermodynamic requirements might supply this "additional information."

Recall from Section 2.4 that the circuit differential equation for Fig. 3-1 is

$$\dot{v}(t) = -\frac{g(v)}{C} - \frac{h(v)}{C}\xi(t).$$

The fourth-order "Kramers-Moyal expanded Fokker-Planck equation" (KME-FPE) for the density ρ is

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial v}\left[\frac{g(v)}{C}\rho\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left[\frac{h^2(v)}{C^2}\rho\right] - \frac{1}{6}\frac{\partial^3}{\partial v^3}\left[a_3(v)\rho\right] + \frac{1}{24}\frac{\partial^4}{\partial v^4}\left[a_4(v)\rho\right] \\
&= \frac{\partial}{\partial v}\left\{\frac{g(v)}{C}\rho + \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{h^2(v)}{C^2}\rho\right] - \frac{1}{6}\frac{\partial^2}{\partial v^2}\left[a_3(v)\rho\right] + \frac{1}{24}\frac{\partial^3}{\partial v^3}\left[a_4(v)\rho\right]\right\}. \quad (3.6)
\end{aligned}
$$

The equilibrium distribution should be Gibbsian:

$$\rho_{eq}(v) = \frac{1}{\sqrt{2\pi kT/C}} \exp\left(-\frac{Cv^2}{2kT}\right).$$

We assume detailed balance (as we did previously for the simple FPE), such that the quantity inside the braces of (3.6) must be identically zero (not just constant with respect to v).

$$0 = \frac{g(v)}{C}\rho_{eq} + \frac{1}{2C^2}\frac{\partial}{\partial v}\left[h^2(v)\rho_{eq}\right] - \frac{1}{6}\frac{\partial^2}{\partial v^2}\left[a_3(v)\rho_{eq}\right] + \frac{1}{24}\frac{\partial^3}{\partial v^3}\left[a_4(v)\rho_{eq}\right]$$

However, even from this equation, it is clear that the extra terms of the expansion cannot be determined by this equation. Conversely, the equilibrium density for the nonlinear Gaussian model is not Gibbsian for any choice of the jump moments. Applying the product rule to the second term on the right hand side will yield a term

$$\frac{1}{2C^2}\left(\frac{\partial h^2(v)}{\partial v}\right)\rho_{eq}(v). \tag{3.7}$$

But the noise model $h^2(v)$ and the jump moments cannot depend on the value of the capacitance. Partial derivatives of the equilibrium distribution will bring down positive powers of the capacitance,

$$\frac{\partial \rho_{eq}(v)}{\partial v} = -\frac{Cv}{kT}\rho_{eq}(v).$$

There is thus no way to cancel out the term (3.7), with the inverse square power of C, except under the same conclusion found in Chapter 2:

$$\frac{\partial h^2(v)}{\partial v} = 0.$$

3.2   The FPE for a Constant-Coefficient SDE

This section derives the simplest Fokker-Planck equation, that corresponding to a stochastic differential equation with constant coefficients. This derivation will serve as a motivation for the much more complicated derivation in Section 3.4 for state-dependent stochastic differential equations.

Consider the equation

$$dx = f\, dt + g\, \xi(t)\, dt, \tag{3.8}$$

where $\xi(t)$ is a unit-variance Gaussian white noise process (note that $\xi(t) = dw(t)$ in the notation of the previous section) and f and g are constants. (They could more generally be functions of time, but this section is trying for the simplest derivation.) This equation is really shorthand for

$$x(t) = x(0) + \int_0^t f\, ds + \int_0^t g\, \xi(s)\, ds. \tag{3.9}$$

Since $\xi(s)$ is a Gaussian random variable for each s and the integration is a linear operation, $x(t)$ must also be Gaussian. Its mean is

$$
\begin{aligned}
E\{x(t)\} &= E\{x(0)\} + E\left\{\int_0^t f\, ds\right\} + E\left\{\int_0^t g\, \xi(s)\, ds\right\} \\
&= x(0) + ft + 0,
\end{aligned}
$$

since $\xi(t)$ is zero-mean. The variance of $x(t)$ is

$$
\begin{aligned}
E\left\{[x(t) - E\{x(t)\}]^2\right\} &= E\left\{\left[x(0) + \int_0^t f\, ds + \int_0^t g\, \xi(s)\, ds - x(0) - ft\right]^2\right\} \\
&= E\left\{\left[\int_0^t g\, \xi(s)ds\right]^2\right\} \\
&= g^2 \int_0^t ds \int_0^t ds'\, E\{\xi(s)\, \xi(s')\} \\
&= g^2\, t.
\end{aligned}
$$

Thus, $x(t)$ is a Gaussian random process with mean ft and variance g^2t. The probability density function for $x(t)$ is

$$\rho(x,t) = \frac{1}{\sqrt{2\pi g^2 t}} \exp\left[-\frac{(x-ft)^2}{2g^2 t}\right]. \tag{3.10}$$

Differentiation yields

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{-1}{2t}\rho(x,t) + \frac{2ft(x-ft)+(x-ft)^2}{2g^2 t^2}\rho(x,t)$$

$$\frac{\partial \rho(x,t)}{\partial x} = -\frac{(x-ft)}{g^2 t}\rho(x,t)$$

$$\frac{\partial^2 \rho(x,t)}{\partial x^2} = \frac{(x-ft)^2}{g^4 t^2}\rho(x,t) - \frac{1}{b^2 t}\rho(x,t).$$

Simple algebra then verifies the statement

$$\frac{\partial \rho(x,t)}{\partial t} = -f\,\frac{\partial \rho(x,t)}{\partial x} + \frac{1}{2}\,g^2\,\frac{\partial^2 \rho(x,t)}{\partial x^2}. \tag{3.11}$$

We have thus shown that, for a stochastic process described by Eq. (3.8), its probability density function evolves according to Eq. (3.11), the Fokker-Planck equation corresponding to the stochastic differential equation. The next section will consider more complicated situations where f and g depend on x and t.

3.3   Stochastic Integrals

In this section, we consider so-called stochastic integrals. The object is to find the proper way to compute the integral

$$\int_a^b w(t)\,dw(t), \tag{3.12}$$

where $w(t)$ is the Brownian motion, with mean 0 and variance t. Note that previous sections have written $\xi(t)$ for $dw(t)$; Gaussian white noise is the name engineers use for the derivative of the Brownian motion. This integral is not well-defined, because w is not of bounded variation. The proper computation of this integral will be important in systems where the amplitude scaling the white noise (g in Eq. (3.8)) depends on the state.

There are two main interpretations of Eq. (3.12): the Itô and Stratonovich. These two integration methods are but two of a continuum of possibilities indexed by a parameter that runs between 0 and 1. Schuss [16] uses λ for this parameter; Arnold [15] uses a (where $a = 1 - \lambda$). We shall use ζ, because λ is much more commonly used as the rate of a Poisson process. The integral is expressed as the limit of a parameterized summation,

$$(\zeta) \int_a^b w(t) \, dw(t) \triangleq \lim_{(P) \, n \to \infty} \sum_{i=0}^{n-1} \left[(1 - \zeta)w(t_i) + \zeta w(t_{i+1})\right] \left[w(t_{i+1}) - w(t_i)\right]. \quad (3.13)$$

where $\lim_{(P) \, n \to \infty}$ means that the summation converges in probability as $n \to \infty$. The fact that $w(t)$ is not of bounded variation implies that the summation converges only in probability but unfortunately not along sample paths. Furthermore, this same lack of smoothness causes the random process to which this function converges in probability to depend on the particular value of ζ, contrary to the more familiar case when $w(t)$ is of bounded variation.

The first subsection is taken from [16], but the remainder of the section was left as exercises in that book.

3.3.1   The Itô Integral

The result of the Itô integral is a martingale. The integral is written as the limit of the summation for $\zeta = 0$,

$$(I) \int_a^b w(t)\, dw(t) \triangleq \lim_{(P)\, n\to\infty} \sum_{i=0}^{n-1} w(t_i)\left[w(t_{i+1}) - w(t_i)\right], \qquad (3.14)$$

where $t_i = a + i[(b-a)/n]$. Let us define

$$I_n \triangleq \sum_{i=0}^{n-1} w(t_i)\left[w(t_{i+1}) - w(t_i)\right],$$

and work on a clever manipulation of the summand.

$$
\begin{aligned}
w(t_i)\left[w(t_{i+1}) - w(t_i)\right] &= w(t_i)w(t_{i+1}) - w^2(t_i) \\
&= w(t_i)w(t_{i+1}) - w^2(t_i) - \frac{1}{2}w^2(t_{i+1}) + \frac{1}{2}w^2(t_{i+1}) \\
&= -\frac{1}{2}\left[w^2(t_{i+1}) - 2w(t_i)w(t_{i+1}) + w^2(t_i)\right] \\
&\qquad + \frac{1}{2}w^2(t_{i+1}) - \frac{1}{2}w^2(t_i) \\
&= -\frac{1}{2}\left[w(t_{i+1}) - w(t_i)\right]^2 + \frac{1}{2}w^2(t_{i+1}) - \frac{1}{2}w^2(t_i) \quad (3.15)
\end{aligned}
$$

Therefore,

$$I_n = \sum_{i=0}^{n-1}\left\{\frac{1}{2}w^2(t_{i+1}) - \frac{1}{2}w^2(t_i) - \frac{1}{2}\left[w(t_{i+1}) - w(t_i)\right]^2\right\}. \qquad (3.16)$$

The first two terms cancel each other out for all intermediate values of i, leaving only the first term for $i = n-1$ and the second term for $i = 0$. Hence,

$$I_n = \frac{1}{2}\left[w^2(b) - w^2(a)\right] - \sum_{i=0}^{n-1}\frac{1}{2}\left[w(t_{i+1}) - w(t_i)\right]^2 = \frac{1}{2}\left[w^2(b) - w^2(a)\right] - \frac{1}{2}\eta_n, \quad (3.17)$$

where

$$\eta_n \triangleq \sum_{i=0}^{n-1}(\delta_i w)^2 \quad \text{and} \quad \delta_i w \triangleq w(t_{i+1}) - w(t_i).$$

Now, each increment $\delta_i w$ of the Brownian motion $w(t)$ has distribution $N(0, t_{i+1} - t_i)$ and is independent of the other increments. Note that the variance is $(t_{i+1} - t_i)$, and standard notation is $N(\mu, \sigma^2)$, whereas Schuss uses $N(\mu, \sigma)$. Therefore,

$$E\{\eta_n\} = E\left\{\sum_{i=0}^{n-1}(\delta_i w)^2\right\} = \sum_{i=0}^{n-1} E\left\{(\delta_i w)^2\right\} = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = b - a. \qquad (3.18)$$

By Chebyshev's inequality,

$$\Pr\left\{\left|\eta_n - E\{\eta_n\}\right| > \epsilon\right\} \leq \frac{\operatorname{Var}\{\eta_n\}}{\epsilon^2}.$$

Again using independence,

$$\operatorname{Var}\{\eta_n\} = \sum_{i=0}^{n-1}\operatorname{Var}\left\{(\delta_i w)^2\right\} = \sum_{i=0}^{n-1} E\left\{(\delta_i w)^4\right\} - E\{(\delta_i w)^2\}^2.$$

Now, since $\delta_i w$ is normally distributed, these expectations are easily calculated:

$$E\left\{(\delta_i w)^4\right\} = 3\sigma^4 = 3(t_{i+1} - t_i)^2$$
$$E\left\{(\delta_i w)^2\right\} = \sigma^2 = (t_{i+1} - t_i),$$

so that the variance of η_n is

$$\operatorname{Var}\{\eta_n\} = \sum_{i=0}^{n-1} 3(t_{i+1} - t_i)^2 - (t_{i+1} - t_i)^2 = \sum_{i=0}^{n-1} 2(t_{i+1} - t_i)^2$$
$$= 2\left(\frac{(b-a)}{n}\right)^2 n = 2\frac{(b-a)^2}{n},$$

because $(t_{i+1} - t_i) = \frac{(b-a)}{n}$ for all i. Returning to Chebyshev's inequality,

$$\Pr\left\{\left|\eta_n - E\{\eta_n\}\right| > \epsilon\right\} \leq \frac{2}{\epsilon^2}\frac{(b-a)^2}{n} \xrightarrow{n \to \infty} 0, \quad \text{for any } \epsilon > 0,$$

which is to say, η_n converges to $E\{\eta_n\} = b - a$ in probability.

Substituting this back into Eq. (3.17),

$$I_n \to \frac{1}{2}\left[w(b) - w(a)\right] - \frac{1}{2}\left[b - a\right]. \tag{3.19}$$

Thus, we have computed the Itô integral

$$(I) \int_a^b w(t)\, dw(t) = \frac{1}{2}\left[w^2(b) - w^2(a)\right] - \frac{1}{2}\left[b - a\right]. \tag{3.20}$$

3.3.2 Backwards Integral

Schuss calls the integral for $\zeta = 1$ the "Backwards integral." This is an unfortunate choice of terminology, because the "backward equation" generally refers to the evolution equation in reverse time for a probability distribution (starting at a given final condition, what is the probability distribution for where the system could have started?). Therefore, when we look for the Fokker-Planck equation for $\zeta = 1$, we must take care to distinguish this from the backward equation.

For this definition of the stochastic integral (3.12), $w(t)$ is evaluated *after* the jump in the summation:

$$(B) \int_a^b w(t)\, dw(t) \overset{\Delta}{=} \lim_{(P)\ n\to\infty} \sum_{i=0}^{n-1} w(t_{i+1})\left[w(t_{i+1}) - w(t_i)\right]. \tag{3.21}$$

In this case, the summand is rearranged as follows:

$$
\begin{aligned}
w(t_{i+1})\left[w(t_{i+1}) - w(t_i)\right] &= w^2(t_{i+1}) - w(t_{i+1})w(t_i) \\
&= w^2(t_{i+1}) - w(t_{i+1})w(t_i) + \frac{1}{2}w^2(t_i) - \frac{1}{2}w^2(t_i) \\
&= \frac{1}{2}\left[w^2(t_{i+1}) - 2w(t_i)w(t_{i+1}) + w^2(t_i)\right] \\
&\quad + \frac{1}{2}w^2(t_{i+1}) - \frac{1}{2}w^2(t_i) \\
&= \frac{1}{2}\left[w(t_{i+1}) - w(t_i)\right]^2 + \frac{1}{2}w^2(t_{i+1}) - \frac{1}{2}w^2(t_i). \quad (3.22)
\end{aligned}
$$

This summand differs from that for the Itô approach (3.15) only in the sign of the first term. Therefore, the rest of the steps are identical to the previous section, except that η_n changes sign. Therefore,

$$(B) \int_a^b w(t) \, dw(t) = \frac{1}{2} \left[w^2(b) - w^2(a) \right] + \frac{1}{2} \left[b - a \right]. \tag{3.23}$$

3.3.3 Stratonovich Integral

Stratonovich [38] defined a different stochastic integral by evaluating $w(t)$ at an average before and after the jump, that is, with $\zeta = 1/2$. In this case, the summand takes on a particularly simple form:

$$(S) \int_a^b w(t) \, dw(t) \;\triangleq\; \lim_{(P) \, n \to \infty} \sum_{i=0}^{n-1} \frac{w(t_{i+1}) + w(t_i)}{2} \left[w(t_{i+1}) - w(t_i) \right] \tag{3.24}$$

$$= \frac{1}{2} \lim_{(P) \, n \to \infty} \sum_{i=0}^{n-1} w^2(t_{i+1}) - w^2(t_i) \tag{3.25}$$

$$= \frac{1}{2} \left[w^2(b) - w^2(a) \right], \tag{3.26}$$

since, as before, the intermediate terms in the summation cancel except the first term for $i = n - 1$ and the second term for $i = 0$.

3.3.4 Arbitrary ζ

Now we are in a position to evaluate the stochastic integral (3.12) for any value of ζ, which specifies when $w(t)$ is evaluated. (The Itô integral has $\zeta = 0$; the Backwards integral has $\zeta = 1$.) Recall

$$(\zeta) \int_a^b w(t) \, dw(t) = \lim_{(P) \, n \to \infty} \sum_{i=0}^{n-1} \left[(1 - \zeta)w(t_i) + \zeta w(t_{i+1}) \right] \left[w(t_{i+1}) - w(t_i) \right]. \tag{3.27}$$

This integral can be computed very quickly by using the results of the Itô and

Backward integrals:

$$\sum_{i=0}^{n-1} \left[(1-\zeta)w(t_i) + \zeta w(t_{i+1})\right] \left[w(t_{i+1}) - w(t_i)\right] \tag{3.28}$$

$$= (1-\zeta)\sum_{i=0}^{n-1} w(t_i)\left[w(t_{i+1}) - w(t_i)\right] + \zeta \sum_{i=0}^{n-1} w(t_{i+1})\left[w(t_{i+1}) - w(t_i)\right]$$

$$\overset{n\to\infty}{\longrightarrow} (1-\zeta)(I)\int_a^b w(t)\,dw(t) + \zeta(B)\int_a^b w(t)\,dw(t)$$

$$= \left[\frac{w^2(b) - w^2(a)}{2}\right] + \left(\zeta - \frac{1}{2}\right)[b-a]. \tag{3.29}$$

The definitions of the Itô and Backwards integrals allowed for a very simple calculation of the ζ-integral. It is possible and only somewhat more difficult to calculate the ζ-integral from the integrals for any two distinct values ζ_1 and ζ_2. Specifically, one may calculate the ζ-integral from the Itô and Stratonovich integrals:

$$\sum_{i=0}^{n-1} \left[(1-\zeta)w(t_i) + \zeta w(t_{i+1})\right]\left[w(t_{i+1}) - w(t_i)\right] \tag{3.30}$$

$$= (1-2\zeta)\sum_{i=0}^{n-1} w(t_i)\left[w(t_{i+1}) - w(t_i)\right]$$

$$+2\zeta \sum_{i=0}^{n-1} \frac{w(t_{i+1}) + w(t_i)}{2}\left[w(t_{i+1}) - w(t_i)\right]$$

$$\overset{n\to\infty}{\longrightarrow} (1-2\zeta)(I)\int_a^b w(t)\,dw(t) + 2\zeta(S)\int_a^b w(t)\,dw(t)$$

$$= \left[\frac{w^2(b) - w^2(a)}{2}\right] + \left(\zeta - \frac{1}{2}\right)[b-a]. \tag{3.31}$$

Since the two usual interpretations are the Itô and Stratonovich integrals, it may prove useful to have this expression. In particular, the Fokker-Planck equation is given in several references in two forms: the Itô and Stratonovich. Therefore, it should be simple to combine them and quickly find the ζ-FPE, without having to investigate the limits of summations and calculate expectations.

3.4   A Dual-Space Derivation of the Fokker-Planck Equation

In this section, the Fokker-Planck equation is derived starting from the stochastic differential equation. We first encountered the derivation in [39]. The arguments are completed by means of characteristic functions in [40].

3.4.1   Single Poisson Counter

Consider the equation

$$x(t) = x(0) + \int_0^t f(x(\sigma), \sigma) \, d\sigma + \int_0^t g(x(\sigma), \sigma) \, dN, \tag{3.32}$$

where N is a Poisson counting process of rate λ, as described in Section 2.5. Brockett [39] states that

> *Definition:* A function $x(\cdot)$ is a solution of Eq. (3.32) in the Itô sense if, on an interval where N is constant, x satisfies $\dot{x} = f(x, t)$ and if N jumps at t_1, x behaves in a neighborhood of t_1 according to the rule
>
> $$\lim_{t \downarrow t_1} x(t) = g(\lim_{t \uparrow t_1} x(t), \ t_1) + \lim_{t \uparrow t_1} x(t)$$
>
> and $x(\cdot)$ is taken to be continuous from the left.

In this case, the "differential form" of Eq. (3.32) is

$$dx = f(x) \, dt + g(x) \, dN. \tag{3.33}$$

(Later in this section, we will consider solutions in other senses than Itô. By evaluating the limit of $g(x(t), t_1)$ in different ways, one derives a ζ-Fokker-Planck equation instead of an Itô FPE.)

Consider some smooth function $\Psi(x)$ with compact support. This is the start of the dual-space argument: in order to prove a statement about a function, we consider its inner product with other functions Ψ. Because of the arbitrariness of Ψ, an equation in its inner product space must also be valid without the inner products.

Itô's rule for jump processes states

$$d\Psi = \frac{d\Psi}{dx} f(x) dt + [\Psi(x + g(x)) - \Psi(x)] \, dN. \tag{3.34}$$

If $g(x)$ were zero in (3.33), the first term simply gives the result of the usual chain rule for differentiation. The second term makes use of the Itô interpretation: if N jumps by 1, then the argument of Ψ jumps from x to $x + g(x)$, therefore Ψ jumps from $\Psi(x)$ to $\Psi(x + g(x))$.

Since $g(x)$ is evaluated before the jump, the martingale property states that dN is independent of $[\Psi(x + g(x)) - \Psi(x)]$. Therefore, the expectation of the product in the second term in (3.34) can be replaced by a product of expectations.

$$
\begin{aligned}
E\left\{[\Psi(x + g(x)) - \Psi(x)] \, dN\right\} &= E\left\{[\Psi(x + g(x)) - \Psi(x)] \, (dN - \lambda \, dt + \lambda \, dt)\right\} \\
&= E\left\{\Psi(x + g(x)) - \Psi(x)\right\} E\left\{dN - \lambda \, dt\right\} \\
&\quad + E\left\{\Psi(x + g(x)) - \Psi(x)\right\} \lambda \, dt \\
&= E\left\{\Psi(x + g(x)) - \Psi(x)\right\} \lambda dt,
\end{aligned}
$$

since $E\left\{dN - \lambda \, dt\right\} = 0$.

Applying the expectation to all the terms in (3.34) and dividing out the dt from the right-hand side yields

$$\frac{d}{dt} E\left\{\Psi\right\} = E\left\{\frac{d\Psi}{dx} f(x)\right\} + E\left\{\Psi(x + g(x)) - \Psi(x)\right\} \lambda. \tag{3.35}$$

Now, assume that there exists a probability distribution $\rho(t, x)$; expectation is calculated by integration against this distribution.

$$
\frac{d}{dt} \int_{-\infty}^{+\infty} \Psi(x)\rho(t, x)dx = \int_{-\infty}^{+\infty} \frac{d\Psi}{dx} f(x)\rho(t, x)dx
$$
$$
+ \int_{-\infty}^{+\infty} [\Psi(x + g(x)) - \Psi(x)] \lambda\rho(t, x)dx \quad (3.36)
$$

Consider the terms individually. For the left-hand side, the order of differentiation and integration may be interchanged:

$$
\frac{d}{dt} \int_{-\infty}^{+\infty} \Psi(x)\rho(t, x) \, dx = \int_{-\infty}^{+\infty} \Psi(x)\frac{\partial}{\partial t}\rho(t, x) \, dx. \quad (3.37)
$$

Integration by parts on the first term on the right-hand side yields

$$
\int_{-\infty}^{+\infty} \frac{d\Psi}{dx} f(x)\rho(t, x) \, dx = \Psi f(x)\rho(t, x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \Psi(x)\frac{\partial}{\partial x}\left[f(x)\rho(t, x)\right] \, dx
$$
$$
= -\int_{-\infty}^{+\infty} \Psi(x)\frac{\partial}{\partial x}\left[f(x)\rho(t, x)\right] \, dx, \quad (3.38)
$$

because Ψ is of compact support. The other term on the right-hand side consists of two subterms,

$$
\int_{-\infty}^{+\infty} [\Psi(x + g(x)) - \Psi(x)] \, \lambda \, \rho(t, x) \, dx
$$
$$
= \int_{-\infty}^{+\infty} \Psi(x + g(x)) \, \lambda \, \rho(t, x) \, dx - \int_{-\infty}^{+\infty} \Psi(x) \, \lambda \, \rho(t, x) \, dx.
$$

The object is to express all the terms of Eq. (3.36) as integrals involving $\Psi(x)$. Let us assume that $\tilde{g}(x) = x + g(x)$ is a one-to-one function, and thus has an inverse. Then a change of variables $y = x + g(x)$ converts the first subterm to

$$
\int_{-\infty}^{+\infty} \Psi(x + g(x)) \, \lambda \, \rho(t, x) \, dx = \int_{-\infty}^{+\infty} \Psi(y) \, \lambda \, \rho(t, \tilde{g}^{-1}(y)) \, J\tilde{g}^{-1}(y) \, dy, \quad (3.39)
$$

where $J\tilde{g}^{-1}(y)$ is the Jacobian (*i.e.*, the absolute value of the determinant) of the transformation $x = \tilde{g}^{-1}(y)$. But y is just a dummy variable, so we can replace it by x to match the other terms.

Using (3.37)-(3.39), (3.36) becomes

$$\int_{-\infty}^{+\infty} \Psi(x) \frac{\partial}{\partial t} \rho(t,x) \, dx \quad =$$

$$-\int_{-\infty}^{+\infty} \Psi(x) \frac{\partial}{\partial x} \left[f(x)\rho(t,x) \right] \, dx$$

$$+\int_{-\infty}^{+\infty} \Psi(x) \, \lambda \, \rho(t, \tilde{g}^{-1}(x)) \, J\tilde{g}^{-1}(x) \, dx$$

$$-\int_{-\infty}^{+\infty} \Psi(x)\lambda\rho(t,x) \, dx. \tag{3.40}$$

Since this must be true for any Ψ (smooth and of compact support), we find that the equation must be true without the integration against Ψ.

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\left[f(x)\rho(t,x) \right] + \lambda\rho(t, \tilde{g}^{-1}(x)) \, J\tilde{g}^{-1}(x) - \lambda\rho(t,x) \tag{3.41}$$

3.4.2   Two Poisson Counters

Consider now consider two independent Poisson counters, with equal rates, but with opposite effects on a random process w_μ governed by

$$dw_\mu = \frac{1}{\sqrt{\mu}} \, (dN_1 - dN_2), \tag{3.42}$$

where now the rates of N_1 and N_2 are $\lambda = \mu/2$. (The drift term $f = 0$.) Obviously,

$$\frac{d}{dt}E\{w_\mu\} = \frac{1}{\sqrt{\mu}} \, (\mu/2 - \mu/2) = 0.$$

Now, if $\Psi = w^2$, Itô's rule for the jump processes states

$$
\begin{aligned}
dw_\mu^2 &= \left[\left(w_\mu + \tfrac{1}{\sqrt{\mu}} \right)^2 - w_\mu^2 \right] dN_1 + \left[\left(w_\mu - \tfrac{1}{\sqrt{\mu}} \right)^2 - w_\mu^2 \right] dN_2 \\
&= \left(\frac{2w_\mu}{\sqrt{\mu}} + \frac{1}{\mu} \right) dN_1 + \left(-\frac{2w_\mu}{\sqrt{\mu}} + \frac{1}{\mu} \right) dN_2 \\
&= \frac{2w_\mu}{\sqrt{\mu}} \left(dN_1 - dN_2 \right) + \frac{1}{\mu} \left(dN_1 + dN_2 \right).
\end{aligned}
$$

Since N_1 and N_2 have the same probabilistic descriptions,

$$
E\{dN_1\} = E\{dN_2\} = (\mu/2)\, dt,
$$

so that

$$
\frac{d}{dt} E\{w_\mu^2\} = 0 + \frac{1}{\mu} \left(\mu/2 + \mu/2 \right) = 1.
$$

Thus, independent of μ, w_μ is a zero-mean process whose variance grows linearly with time. If w_μ starts deterministically at the origin,

$$
E\{w_\mu^2(t)\} = t.
$$

In fact, there is an easier way to obtain this result. Since the drift term $f(x,t)$ does not appear in Eq. (3.42), and the jump height $g(x,t)$ is independent of x, the equation is equivalent to

$$
w_\mu(t) = \frac{1}{\sqrt{\mu}} \left[N_1(t) - N_2(t) \right].
$$

From this equation, it is immediate that

$$
E\{w_\mu(t)\} = 0 \quad \text{and} \quad E\{w_\mu^2(t)\} = t.
$$

In the limit as $\mu \to \infty$, the rates of the jump processes N_1 and N_2 are becoming

very large, but the length of the step taken at each jump is vanishing. The first and second moments match those for the Brownian motion for all μ. Further calculations show that the higher moments converge as $\mu \to \infty$ to the moments of the Brownian motion. It remains to be shown that the process we obtain in fact *is* the Brownian motion. For this, we use the result cited in [40], namely that convergence in distribution follows from convergence of characteristic functions.

The characteristic function of a Gaussian (or normal) random process with mean 0 and variance t is

$$\phi(u; t) = \exp\left(-\frac{1}{2}u^2 t\right).$$

The characteristic function of a Poisson random process $N(t)$ of rate λ is

$$\phi_N(u; t) = E\left\{\exp\left(juN(t)\right)\right\} = \exp\left[\lambda t\left(\exp(ju) - 1\right)\right],$$

where $j = \sqrt{-1}$. Using properties of the characteristic function (scaling the step heights and recognizing that the characteristic function of the sum of two independent random variables is the product of their characteristic functions), the characteristic function of w_μ is seen to be

$$
\begin{aligned}
\phi_\mu(u; t) &= \exp\left[\frac{\mu}{2}t\left(\exp(ju/\sqrt{\mu}) - 1\right)\right]\exp\left[\frac{\mu}{2}t\left(\exp(-ju/\sqrt{\mu}) - 1\right)\right] \\
&= \exp\left[\frac{\mu}{2}t\left(\exp(ju/\sqrt{\mu}) + \exp(-ju/\sqrt{\mu}) - 2\right)\right].
\end{aligned}
$$

As $\mu \to \infty$, the inside exponentials may be expanded

$$\exp(ju/\sqrt{\mu}) = 1 + \frac{ju}{\sqrt{\mu}} - \frac{u^2}{2\mu} + o(1/\mu).$$

Then,

$$\phi_\mu(u; t) = \exp\left[\frac{\mu}{2}t\left(1 + \frac{ju}{\sqrt{\mu}} - \frac{u^2}{2\mu} + o(1/\mu) + 1 - \frac{ju}{\sqrt{\mu}} - \frac{u^2}{2\mu} + o(1/\mu) - 2\right)\right]$$

$$= \ \exp\left[-\frac{u^2}{2}t + \frac{\mu t}{2} o(1/\mu)\right], \tag{3.43}$$

where $o(1/\mu)$ denotes a term such that $\mu\, o(1/\mu) \to 0$ as $\mu \to \infty$. Since the characteristic function of w_μ converges to the characteristic function of Brownian motion, we conclude that w_μ converges in distribution to the Brownian motion.

Consider now the process z governed by

$$dz = \frac{1}{\mu}\, (dN_1 + dN_2)\,.$$

Its mean evolves as

$$\frac{d}{dt}Ez = 1 \quad \Rightarrow \quad Ez(t) = t + Ez(0).$$

Itô's rule gives for the square of z

$$dz^2 = \left[(z + 1/\mu)^2 - z^2\right](dN_1 + dN_2) = \left(\frac{2z}{\mu} + \frac{1}{\mu^2}\right)(dN_1 + dN_2),$$

so that the mean-square value evolves as

$$\begin{aligned}
\frac{d}{dt}Ez^2 &= \left(\frac{2Ez}{\mu} + \frac{1}{\mu^2}\right)(\mu/2 + \mu/2) = 2Ez + \frac{1}{\mu} \\
&= 2t + 2Ez(0) + \frac{1}{\mu}.
\end{aligned}$$

Then

$$Ez^2(t) = t^2 + 2tEz(0) + \frac{t}{\mu} + Ez^2(0).$$

Supposing that $z(0) = 0$ with probability 1, $Ez(0) = Ez^2(0) = 0$. Then considering the limit as $\mu \to \infty$, the variance approaches zero:

$$Ez^2(t) - (Ez(t))^2 = t/\mu \to 0.$$

We then identify

$$dz = dt \tag{3.44}$$

in the limit $\mu \to \infty$.

3.4.3   Itô Rule for Brownian Motion

We can use the calculations of the previous section to obtain a new Itô rule for a stochastic process driven by Brownian motion. We will approximate

$$dx = f(x)\,dt + g(x)\,dw$$

by the jump-process-driven differential

$$dx = f(x)\,dt + g(x)\frac{1}{\sqrt{\mu}}\,(dN_1 - dN_2)\,, \tag{3.45}$$

(where the rates of N_1 and N_2 are $\mu/2$) and consider the limit $\mu \to \infty$.

Again, considering an arbitrary $\Psi(x)$ (smooth and of compact support), Itô's rule for jump processes yields

$$d\Psi = \frac{d\Psi}{dx}f(x)\,dt + \left[\Psi\left(x + g(x)\tfrac{1}{\sqrt{\mu}}\right) - \Psi(x)\right]dN_1 + \left[\Psi\left(x - g(x)\tfrac{1}{\sqrt{\mu}}\right) - \Psi(x)\right]dN_2. \tag{3.46}$$

Under the further assumption that Ψ is twice differentiable, the Taylor expansion of $\Psi(x + \delta)$ about x for small $\delta = \pm g(x)/\sqrt{\mu}$ is

$$
\begin{aligned}
d\Psi \;=\; & \frac{d\Psi}{dx}f(x)\,dt \\
& + \left[\Psi(x) + \frac{d\Psi}{dx}\frac{g(x)}{\sqrt{\mu}} + \frac{1}{2}\frac{d^2\Psi}{dx^2}\frac{g^2(x)}{\mu} + o(1/\mu) - \Psi(x)\right]dN_1 \\
& + \left[\Psi(x) - \frac{d\Psi}{dx}\frac{g(x)}{\sqrt{\mu}} + \frac{1}{2}\frac{d^2\Psi}{dx^2}\frac{g^2(x)}{\mu} + o(1/\mu) - \Psi(x)\right]dN_2
\end{aligned}
$$

$$= \frac{d\Psi}{dx} f(x)\, dt + \frac{d\Psi}{dx} \frac{g(x)}{\sqrt{\mu}} (dN_1 - dN_2) + \frac{1}{2} \frac{d^2\Psi}{dx^2} \frac{g^2(x)}{\mu} (dN_1 + dN_2)$$
$$+ o(1/\mu)\, dN_1 + o(1/\mu)\, dN_2,$$

where $o(1/\mu)$ denotes terms such that $\mu\, o(1/\mu) \to 0$ as $\mu \to \infty$. Since $E\{dN_1\}$ and $E\{dN_2\}$ are in fact of order μ, the last two terms vanish as $\mu \to \infty$. Using the identifications in the previous subsection for w and z in the same limit, we have derived a new Itô's rule for a stochastic process driven by Brownian motion:

$$d\Psi = \frac{d\Psi}{dx} f(x)\, dt + \frac{d\Psi}{dx} g(x)\, dw + \frac{1}{2} \frac{d^2\Psi}{dx^2} g^2(x)\, dt. \tag{3.47}$$

3.4.4 Fokker-Planck Equation

Using this new Itô rule, we may now derive the forward equation for a stochastic process driven by Brownian motion, that is, the Fokker-Planck equation. Consider the stochastic process governed by

$$dx = f(x)\, dt + g(x)\, dw. \tag{3.48}$$

For Ψ twice-differentiable and of compact support,

$$d\Psi = \frac{d\Psi}{dx} f(x)\, dt + \frac{d\Psi}{dx} g(x)\, dw + \frac{1}{2} \frac{d^2\Psi}{dx^2} g^2(x)\, dt.$$

Now, dw has expectation 0 and is independent of x by the martingale property, so that

$$\frac{d}{dt} E\{\Psi\} = E\left\{ \frac{d\Psi}{dx} f(x) \right\} + \frac{1}{2} E\left\{ \frac{d^2\Psi}{dx^2} g^2(x) \right\}. \tag{3.49}$$

Assuming that there exists a suitable $\rho(t,x)$, the expectation may be expressed as an integral, as before.

$$\frac{d}{dt} \int_{-\infty}^{+\infty} \rho(t,x)\Psi\, dx = \int_{-\infty}^{+\infty} \rho(t,x) \frac{d\Psi}{dx} f(x)\, dx + \frac{1}{2} \int_{-\infty}^{+\infty} \rho(t,x) \frac{d^2\Psi}{dx^2} g^2(x)\, dx \tag{3.50}$$

Integration by parts yields

$$\int_{-\infty}^{+\infty} \Psi(x)\frac{\partial}{\partial t}\rho(t,x)\ dx = -\int_{-\infty}^{+\infty} \Psi(x)\frac{d}{dx}\Big[f(x)\rho(t,x)\Big]\ dx$$
$$+ \ \frac{1}{2}\int_{-\infty}^{+\infty} \Psi(x)\frac{\partial}{\partial x^2}\Big[\rho(t,x)g^2(x)\Big]\ dx, \qquad (3.51)$$

because Ψ has compact support). Since expression must hold for every Ψ, it must also hold without the integration against Ψ.

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\Big[f(x)\rho(t,x)\Big] + \frac{1}{2}\frac{\partial}{\partial x^2}\Big[\rho(t,x)g^2(x)\Big]\ dx \qquad (3.52)$$

This is the (Itô-form) Fokker-Planck equation corresponding to the stochastic differential equation (3.48).

3.4.5   Non-Itô Fokker-Planck Equations

As there are different interpretations of the stochastic integral, as defined in Section 3.3, there are also different forms of the Fokker-Planck equation.

In particular, for the Stratonovich interpretation of the stochastic differential equation

$$dx = f(x)\ dt + g(x)\ dw, \qquad (3.53)$$

the corresponding Fokker-Planck equation is [19, 41]

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\Big[f(x)\rho(t,x) + \frac{1}{2}g(x)\frac{dg(x)}{dx}\rho(t,x)\Big] + \frac{1}{2}\frac{\partial}{\partial x^2}\Big[\rho(t,x)g^2(x)\Big]\ dx. \quad (3.54)$$

Note that this equation could also be obtained in the Itô interpretation for the stochastic differential equation

$$dx = \Big[f(x) + \frac{1}{2}g(x)\frac{dg(x)}{dx}\Big]\ dt + g(x)\ dw.$$

The term $\frac{1}{2}g(x)g'(x)$ is called the "Zakai-Wong correction term" in [16]. Other authors [19, 41] have also given the transformation between Itô and Stratonovich stochastic differentials. The correction term is sometimes called "spurious drift" [20], since it adds a contribution to the drift term of the (Itô) equation, based on the form of the noise. Note that the correction vanishes in the case that $g(x)$ is a constant.

For the case of the "Backward integral," there is yet another Fokker-Planck equation, derived in [42] using the same methods as above:

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\left[f(x)\rho(t,x) + g(x)\frac{dg(x)}{dx}\rho(t,x)\right] + \frac{1}{2}\frac{\partial}{\partial x^2}\left[\rho(t,x)g^2(x)\right]\,dx. \quad (3.55)$$

In fact, for any value of ζ, it can be shown [43] that the ζ-Fokker-Planck equation for Eq. (3.53) is

$$\frac{\partial}{\partial t}\rho(t,x) = -\frac{\partial}{\partial x}\left[f(x)\rho(t,x) + \zeta\, g(x)\frac{dg(x)}{dx}\rho(t,x)\right] + \frac{1}{2}\frac{\partial}{\partial x^2}\left[\rho(t,x)g^2(x)\right]\,dx. \quad (3.56)$$

3.5   Brownian Motion and Jumps

Consider the stochastic differential equation driven by both white noise and a Poisson counter,

$$dx = f(x)\,dt + g(x)\,dN + h(x)\,dw, \qquad (3.57)$$

where dw is Gaussian white noise, understood as the limit of the two opposed Poisson counters, and N is a Poisson counter of rate λ.

It is a simple extension of the derivations in the previous section to show that the forward equation for the probability distribution in this case is

$$\begin{aligned}
\frac{\partial\rho(t,x)}{\partial t} &= -\frac{\partial}{\partial x}\left[f(x)\,\rho(t,x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[h^2(x)\,\rho(t,x)\right]\\
&\quad +\lambda\,\rho(t,\tilde{g}^{-1}(x))\,J\tilde{g}^{-1}(x) - \lambda\,\rho(t,x). \qquad (3.58)
\end{aligned}$$

It will be useful for the following chapter to derive the forward equation for a special vector case:

$$d\mathbf{x} = \mathbf{a}(\mathbf{x})\, dt + \mathbf{b}\, \mathbf{dw} + \sum_{i=1}^{m} \mathbf{g}_i\, dN_i(\lambda_i(\mathbf{x})), \tag{3.59}$$

where $\mathbf{w}$ is a vector of Brownian motions. The number m of Poisson counters need not be the same as the dimension n of the vector $\mathbf{x}$. In this special case, the coefficients of both the Brownian motions and the Poisson counter are constants. Therefore, $\tilde{\mathbf{g}} = \mathbf{x} + \mathbf{g}$ and $\mathbf{J}\tilde{\mathbf{g}}^{-1}(\mathbf{x})$ is the identity matrix. However, the rates $\lambda_i(\mathbf{x})$ of the Poisson counters are not constant, but instead depend on a function of the present state. The existence and uniqueness of solutions $\mathbf{x}(t)$ to this equation will be addressed in Appendix C.

Itô's rule for this SDE yields

$$
\begin{aligned}
d\Psi(\mathbf{x}) \;=\;& \left(\nabla\Psi(\mathbf{x})\right)\mathbf{a}(\mathbf{x})\, dt + \left(\nabla\Psi(\mathbf{x})\right)\mathbf{b}\, \mathbf{dw} + \left(\nabla^2\Psi(\mathbf{x})\right)\mathbf{b}\mathbf{b}^T \\
& + \sum_{i=1}^{n} \left[\Psi(\mathbf{x} + \mathbf{g}_i(\mathbf{x})) - \Psi(\mathbf{x})\right] dN_i(\lambda_i(\mathbf{x})).
\end{aligned}
$$

Using the Itô interpretation means that $\mathbf{x}$ is a martingale, and further, $\mathbf{x}$ is independent of $\mathbf{dw}$, which is a zero-mean random variable. Taking expectations of this last equation does require one further trick, however.

$$
\begin{aligned}
& E\Big\{\left[\Psi(\mathbf{x} + \mathbf{g}_i(\mathbf{x})) - \Psi(\mathbf{x})\right] dN_i(\lambda_i(\mathbf{x}))\Big\} \\
=\;& E\Big\{\left[\Psi(\mathbf{x} + \mathbf{g}_i(\mathbf{x})) - \Psi(\mathbf{x})\right]\Big(dN_i(\lambda_i(\mathbf{x})) - \lambda_i(\mathbf{x})\, dt + \lambda_i(\mathbf{x})\, dt\Big)\Big\}
\end{aligned}
$$

Conditioned on the rate λ_i, the Poisson counters have independent increments. Therefore, conditioned on the rate, the increment described by $dN_i(\lambda_i(\mathbf{x})) - \lambda_i(\mathbf{x})\, dt$ is independent of previous jumps in the process. Further, for any rate $\lambda_i(\mathbf{x})$, this increment

is zero mean,

$$E\left\{dN_i(\lambda_i(\mathbf{x})) - \lambda_i(\mathbf{x})\,dt\right\} = 0.$$

Since this is true for any rate $\lambda_i(\mathbf{x})$, we claim that

$$E\left\{\left[\Psi(\mathbf{x} + \mathbf{g}_i(\mathbf{x})) - \Psi(\mathbf{x})\right]dN_i(\lambda_i(\mathbf{x}))\right\} = E\left\{\left[\Psi(\mathbf{x} + \mathbf{g}_i(\mathbf{x})) - \Psi(\mathbf{x})\right]\lambda_i(\mathbf{x})\,dt\right\}.$$

(3.60)

Note that $\lambda_i(\mathbf{x})$ is inside the expectation.

Following the procedures of the previous section, we arrive at the forward equation for the probability distribution for the system driven by both Brownian motion and Poisson counters:

$$
\begin{aligned}
\frac{\partial \rho(t, \mathbf{x})}{\partial t} &= -\nabla^T\left[\mathbf{a}(\mathbf{x})\,\rho(t, \mathbf{x})\right] + \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(\mathbf{b}\mathbf{b}^T\right)_{ij}\rho(t, \mathbf{x}) \\
&\quad + \sum_{i=1}^{m}\lambda_i(\mathbf{x} - \mathbf{g}_i)\rho(t, \mathbf{x} - \mathbf{g}_i) - \sum_{i=1}^{m}\lambda_i(\mathbf{x})\rho(t, \mathbf{x}).
\end{aligned}
$$

(3.61)

3.6   Interpretations of the Stochastic Integral for the Nonlinear Gaussian Model

Gaussian white noise of unlimited bandwidth is an idealization of the derivative of Brownian motion $w(t)$. Though $\frac{dw}{d\tau}$ does not exist, (2.15) is really shorthand for

$$
\begin{aligned}
v(t) &= v(0) - \frac{1}{C}\int_0^t g_T\left(v(\tau)\right)d\tau + \frac{1}{C}\int_0^t h_T\left(v(\tau)\right)\frac{dw}{d\tau}d\tau \\
&= v(0) - \frac{1}{C}\int_0^t g_T\left(v(\tau)\right)d\tau + \frac{1}{C}\int_0^t h_T\left(v(\tau)\right)dw(\tau).
\end{aligned}
$$

The second line, in which $\frac{dw}{d\tau}$ does not appear, is almost a rigorous statement of the meaning of (2.15), since $v(t)$ and $w(t)$ are continuous functions. But one ambiguity remains: the interpretation of the second integral in terms of the stochastic integrals

defined in Section 3.3. Although that section dealt with the integral

$$\int w(t)\, dw(t),$$

the same problem arises for any function of Brownian motion,

$$\int f(x)\, dw(t),$$

when x is a random process driven by the white noise $dw(t)$.

For Eq. (2.15), the stochastic integral may be expressed as the summation parameterized by ζ,

$$
\begin{aligned}
(\zeta) \int_0^t & h_T(v(\tau))\, dw(\tau) \\
& \overset{\triangle}{=} \lim_{(P)\, n\to\infty} \sum_{i=0}^{n-1} \left[(1-\zeta)\, h_T\left(v\left(\frac{it}{n} \right) \right) + \zeta\, h_T\left(v\left(\frac{(i+1)t}{n} \right) \right) \right] \\
& \qquad\qquad \times \left[w\left(\frac{(i+1)t}{n} \right) - w\left(\frac{it}{n} \right) \right],
\end{aligned}
\tag{3.62}
$$

as shown in Section 3.3 for a simpler integral. The literature is primarily concerned with two interpretations for the above equation: the Itô or stochastic integral ($\zeta = 0$) and the Stratonovich integral ($\zeta = 1/2$). The Itô approach yields a non-anticipating martingale [16]. The Stratonovich approach is obtained by considering mathematical limits of idealized physical systems. Rationalization for the other values of ζ is not clear.

When the stochastic differential equation (2.15) is interpreted in a more general sense for any ζ as an integral equation

$$v(t) = v(0) - \frac{1}{C} \int_0^t g_T\left(v(\tau) \right) d\tau + \frac{1}{C}(\zeta) \int_0^t h_T\left(v(\tau) \right) dw(\tau),$$

and when the functions g_T and h_T and their derivatives are continuous and satisfy

Lipschitz conditions (found in [30]), it can be shown [42, 43] that the corresponding Fokker-Planck equation is

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial v} \left\{ \frac{g_T(v)}{C} \rho(v,t) - \frac{\zeta h_T(v)\rho(v,t)}{C^2} \frac{\partial}{\partial v} h_T(v) + \frac{1}{2} \frac{\partial}{\partial v} \left[\frac{h_T^2(v)}{C^2} \rho(v,t) \right] \right\}, \quad (3.63)$$

which simplifies to (2.16) or (2.17), respectively, in the Itô and Stratonovich cases.

Although the rational and physical justification for other values of ζ are unclear, one specific value weakens the conclusion in Section 2.4. For $\zeta = 1$, which yields a "Backwards integral" [16], there is a second form of solution to (2.19):

$$h_T^2(v) = 2kT \frac{g_T(v)}{v}, \quad (3.64)$$

a unique noise amplitude determined solely by the resistor constitutive relation, again independent of C. Note that it reduces to the Nyquist-Johnson model in the case of a linear resistor.

3.6.1   Entropy for $\zeta = 1$

In Chapter 2, we did not check the entropy rate of the nonlinear Gaussian model for compliance with Thermodynamic Requirement #3, since the model did not satisfy the second Requirement of the Gibbs distribution. For completeness, we should check the entropy for the $\zeta = 1$ Fokker-Planck equation.

The nonlinear Gaussian model is constructed in voltage rather than charge. Corresponding to Eq. (2.8), the capacitor entropy rate in voltage is

$$\dot{S}_C = -k \int_{-\infty}^{+\infty} \dot{\rho}_v \ln \rho_v \, dv. \quad (3.65)$$

The reservoir entropy rate, corresponding to Eq. (2.9) is

$$\dot{S}_R = -\frac{1}{T} \int_{-\infty}^{+\infty} E_C(v) \, \dot{\rho}_v \, dv. \tag{3.66}$$

The total entropy rate is the sum of the last two equations, but we will substitute Eq. (3.63) for $\dot{\rho}_v$ and then integrate by parts.

$$
\begin{aligned}
\dot{S}_{tot} &= -\int_{-\infty}^{+\infty} \left(k \ln \rho_v + \frac{E_C(v)}{T} \right) \dot{\rho}_v \, dv \\
&= -\int_{-\infty}^{+\infty} \left(k \ln \rho_v + \frac{Cv^2}{2T} \right) \frac{\partial}{\partial v} \left\{ \frac{g_T(v)}{C} \rho_v - \frac{\zeta h_T(v)\rho_v}{C^2} \frac{\partial}{\partial v} h_T(v) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{1}{2} \frac{\partial}{\partial v} \left[\frac{h_T^2(v)}{C^2} \rho_v \right] \right\} \, dv \\
&= \int_{-\infty}^{+\infty} \left(k \frac{1}{\rho_v} \frac{\partial \rho_v}{\partial v} + \frac{Cv}{T} \right) \left\{ \frac{g_T(v)}{C} \rho_v - \frac{\zeta h_T(v)\rho_v}{C^2} \frac{\partial}{\partial v} h_T(v) \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \frac{h_T(v)}{C^2} \frac{\rho_v}{C^2} \frac{\partial h_T(v)}{\partial v} + \frac{h_T^2(v)}{C^2} \frac{\partial \rho_v}{\partial v} \right\} \, dv \\
&= \int_{-\infty}^{+\infty} \left(k \frac{1}{\rho_v} \frac{\partial \rho_v}{\partial v} + \frac{Cv}{T} \right) \left\{ \frac{g_T(v)}{C} \rho_v + (1-\zeta) h_T(v) \frac{h_T(v)\rho_v}{C^2} \frac{\partial}{\partial v} h_T(v) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{h_T^2(v)}{C^2} \frac{\partial \rho_v}{\partial v} \right\} \, dv
\end{aligned}
$$

It is not obvious whether this is nonnegative for arbitrary values of ζ. Since the nonlinear Gaussian model has the correct equilibrium density only when $\zeta = 1$ and $h^2(v) = 2kTg(v)/v$, let us proceed with that value of ζ.

$$
\begin{aligned}
\dot{S}_{tot} &= \int_{-\infty}^{+\infty} \left(k \frac{1}{\rho_v} \frac{\partial \rho_v}{\partial v} + \frac{Cv}{T} \right) \left\{ \frac{g_T(v)}{C} \rho_v + \frac{2kTg(v)}{v\,C^2} \frac{\partial \rho_v}{\partial v} \right\} \, dv \\
&= \int_{-\infty}^{+\infty} \left(k \frac{1}{\rho_v} \frac{\partial \rho_v}{\partial v} + \frac{Cv}{T} \right) \left[\frac{g(v)}{v} \frac{T}{C^2} \rho_v \right] \left(\frac{Cv}{T} + \frac{k}{\rho_v} \frac{\partial \rho_v}{\partial v} \right) \, dv \\
&= \int_{-\infty}^{+\infty} \left[\frac{g(v)}{v} \frac{T}{C^2} \rho_v \right] \left(\frac{Cv}{T} + \frac{k}{\rho_v} \frac{\partial \rho_v}{\partial v} \right)^2 \, dv \tag{3.67}
\end{aligned}
$$

Since the device is passive, $g(v)/v > 0$ for $v \neq 0$, and hence the integrand is always
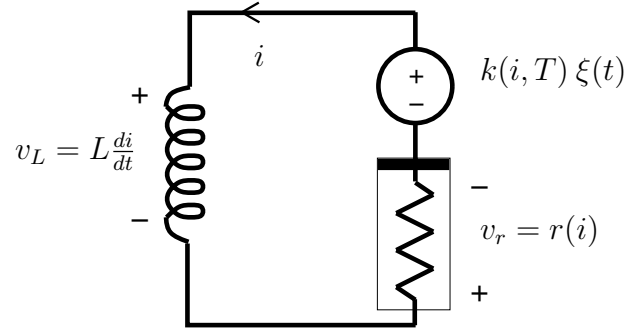
Figure 3-2: Voltage-noise source with inductor

nonnegative. Thermodynamic Requirement #3 is satisfied for the nonlinear Gaussian model with $\zeta = 1$.

3.6.2   Nonlinear Gaussian Model with an Inductor

Unfortunately, testing the Thermodynamic Requirements for a capacitor load is not sufficient to prove that a noise model is valid. If one instead considers the nonlinear resistance in a circuit with an inductor, one gets an incompatible variance for the white noise voltage.

The differential equation governing the circuit in Fig. 3-2 is

$$\frac{di}{dt} = -\frac{r(i)}{L} + \frac{k(i,T)}{L}\xi(t). \tag{3.68}$$

The Fokker-Planck equation with the Zakai-Wong correction states

$$\frac{\partial \rho(i,t)}{\partial t} = \frac{\partial}{\partial i}\left\{\frac{r(i)}{L}\rho(i,t) - \frac{\zeta k(i,T)}{L^2}\rho(i,t)\frac{\partial k(i,T)}{\partial i} + \frac{1}{2}\frac{\partial}{\partial i}\left[\frac{k^2(i,T)}{L^2}\rho(i,t)\right]\right\}. \tag{3.69}$$

Thermodynamic Requirement #2 states that the the equilibrium density is Gibbs

distribution in terms of inductor energy as a function of current:

$$\rho_{eq}(i) = \frac{\exp\left(-\frac{Li^2}{2kT}\right)}{\sqrt{2\pi kT/L}}.$$

Again, the term in braces in (3.69) must vanish identically at equilibrium:

$$0 = \frac{r(i)}{L} - \frac{\zeta}{2L^2}\frac{\partial k^2(i,T)}{\partial i} + \frac{1}{2L^2}\frac{\partial k^2(i,T)}{\partial i} + \frac{k^2(i,T)}{2L^2}\left(\frac{-Li}{kT}\right).$$

Corresponding to (2.19), we get

$$(\zeta - 1)\frac{\partial k^2(i,T)}{\partial i} = 2Lr(i) - \frac{Li}{kT}k^2(i,T).$$

The voltage noise source cannot know the value of the inductor, so both sides must equal zero. From the right hand side, we get

$$k^2(i,T) = \frac{2kTr(i)}{i}. \tag{3.70}$$

If ζ is arbitrary, the resistor must be linear, as before, and

$$\frac{\partial k^2(i,T)}{\partial i} = 0 \Rightarrow k^2(i,T) = 2kTR.$$

If $\zeta = 1$, we are again considering the Backward integral, and we get a unique noise amplitude (3.70) for each constitutive relation $r(i)$ for the nonlinear resistor. Similar calculations can be performed on the Hamiltonian system consisting of a linear capacitor and inductor with the nonlinear resistance, either all in parallel and with a current noise source, or all in series with a voltage noise source. The state space is now two dimensional, so the calculations are more complex, but the result is the same: for a voltage noise source, one obtains (3.64) and for a current noise source one obtains (3.70). However, this voltage noise amplitude does not correspond to the
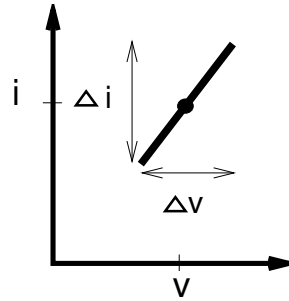
Figure 3-3: Fluctuations

result obtained for the parallel current noise course. Consider biasing the device at
a certain voltage and current combination. If we bias at constant voltage we observe
current fluctuations. If we bias at constant current, we see voltage fluctuations, which
in turn give rise to current fluctuations through the resistor. Since the internal state
of the nonlinear device is the same whether we specify the voltage or current (for
a monotone $v - i$ curve), we require these fluctuations to be equivalent. Consider
Fig. 3-3. The equations corresponding to this requirement are as follows:

$$
\begin{aligned}
\Delta v \, \frac{di}{dv} &= \Delta i \Rightarrow \\
k(i,T) \frac{di}{dv} = h(v,T) &\Rightarrow \sqrt{2kT\frac{r(i)}{i}} \, \frac{di}{dv} = \sqrt{2kT\frac{g(v)}{v}} \Rightarrow \\
\frac{di}{dv} &= \frac{i}{v}.
\end{aligned}
$$

The only device for which $di/dv = i/v$ around every operating point is the linear
resistor. We interpret this to mean that the Gaussian noise model does *not* give us a
correct noise amplitude for *any* nonlinear device.

3.7   Convergence of Poisson to Gaussian

This section is motivated by the appearances of the formula

$$S_{ii}(\omega; T, V) = \frac{2kT\, g(V)}{V} \tag{3.71}$$

in two different situations.

First, for a nonlinear Gaussian model, if the white noise is scaled by the amplitude

$$h_T(v) = \sqrt{2kTg(v)/v}$$

and the $\zeta = 1$ Fokker-Planck equation is used, then the model satisfies the thermo-dynamic requirements and has the power spectral density (3.71) when the voltage is held constant.

Second, in the Poisson model, where

$$S_{ii}(\omega; T, V) = \frac{e\, g(V)}{\tanh(eV/kT)},$$

the limit as the electron size $e \to 0$ yields again (3.71) by l'Hôpital's rule. Section 2.6 showed this to be true for a linear resistor, but it holds more generally.

This "coincidence" will be investigated in two ways: first, from the limiting behavior of the charge random process at a fixed voltage V; and second, from the limiting behavior of the forward equation for the probability density. The fact that Brownian motion was obtained as the limit of Poisson counters further encourages us in this approach.

3.7.1   The Random Process

The goal of this section is to investigate the random process of the total charge emitted by the Poisson device. Does it become a Brownian motion in the limit that

the electron charge goes to zero?

The Poisson model has two oppositely directed Poisson counters with rates

$$\lambda_f = f_T(v) \qquad \lambda_r = r_T(v),$$

which depend on the temperature and instantaneous applied voltage. The device constitutive relation is

$$g(v) = \overline{i(v)} = e\left[f_T(v) - r_T(v)\right]. \tag{3.72}$$

The thermodynamic requirement is

$$\frac{f_T(v)}{r_T(v)} = \exp(ev/kT) = \exp(v/v_T). \tag{3.73}$$

By using (3.73), one may solve for the forward and reverse rates in terms of the constitutive relation:

$$r_T(v) \;=\; \frac{g(v)}{e\left[\exp(ev/kT) - 1\right]} \tag{3.74}$$

$$f_T(v) \;=\; \frac{g(v)}{e\left[1 - \exp(-v/v_T)\right]}. \tag{3.75}$$

For small e, one can expand the exponentials ($e^x = 1 + x + x^2/2 + \ldots$) and the reciprocal ($1/(1 + x) = 1 - x + x^2 + \ldots$) in both of these last two equations.

$$
\begin{aligned}
r_T(v) \;&=\; \frac{g(v)}{e\left[(1 + ev/kT + (ev/kT)^2/2 + \ldots) - 1\right]} \\
&=\; \frac{g(v)}{e^2}\frac{kT}{v}\left[1 - \frac{1}{2}\frac{ev}{kT} + \frac{1}{12}\left(\frac{ev}{kT}\right)^2 - \frac{1}{720}\left(\frac{ev}{kT}\right)^4 + \ldots\right] \\
f_T(v) \;&=\; \frac{g(v)}{e\left[1 - (1 - ev/kT + (ev/kT)^2/2 + \ldots)\right]} \\
&=\; \frac{g(v)}{e^2}\frac{kT}{v}\left[1 + \frac{1}{2}\frac{ev}{kT} + \frac{1}{12}\left(\frac{ev}{kT}\right)^2 - \frac{1}{720}\left(\frac{ev}{kT}\right)^4 + \ldots\right]
\end{aligned}
$$

Note that

$$f_T(v) - r_T(v) = \frac{g(v)}{e^2} \frac{kT}{v} \left[\frac{ev}{kT} \right] = \frac{g(v)}{e},$$

because the other terms cancel exactly.

However,

$$f_T(v) + r_T(v) = \frac{g(v)}{e^2} \frac{kT}{v} \left[2 + \frac{2}{12} \left(\frac{ev}{kT} \right)^2 - \frac{2}{720} \left(\frac{ev}{kT} \right)^4 + \ldots \right],$$

so that

$$e^2 \left[f_T(v) + r_T(v) \right] = \frac{2kT}{v} \frac{g(v)}{v} \tag{3.76}$$

is only exact for $e = 0$.

Now let us consider, at a fixed voltage V, the random process defined by

$$i = dq = e \left[dN_f - dN_r \right], \tag{3.77}$$

where N_f and N_r are a Poisson processes with rates $f_T(V)$ and $r_T(V)$, respectively. Then,

$$\frac{d}{dt} E\{q\} = e \left[f_T(V) - r_T(V) \right] = g(v) \tag{3.78}$$

so that

$$E\{q(t)\} = E\{q(0)\} + g(V)t. \tag{3.79}$$

Since, for fixed V, the processes N_f and N_r are independent, one adds their variances (and scales by e^2) to find the variance of $q = e[N_f - N_r]$.

$$\begin{aligned} \mathrm{Var}\{q(t)\} &= \mathrm{Var}\{q(0)\} + e^2 \left[f_T(V) + r_T(V) \right] t \tag{3.80} \\ &= \mathrm{Var}\{q(0)\} + 2kT \frac{g(V)}{V} \left[1 + \frac{1}{12} \left(\frac{eV}{kT} \right)^2 - \frac{1}{720} \left(\frac{eV}{kT} \right)^4 + \ldots \right] t \tag{3.81} \end{aligned}$$

The process $M^e(t) = q(t) - E\{q(t)\}$ is zero mean, and has a variance that grows linearly with t, for all values of e. Because the Poisson counters have independent

increments, $w(t)$ does as well. Unfortunately, Brownian motion is continuous, whereas $M^e(t)$ is not continuous for any $e > 0$, so that we need the following martingale convergence theorem from Karr [44] to show that $M^e(t)$ has a limit as $e \to 0$ that is a continuous martingale.

> *Theorem 5.10* [44]. Let $M^1, M^2, \ldots$ be mean zero square integrable martingales and let v be a continuous nondecreasing function on $[0, 1]$ with $v_0 = 0$. Suppose that
>
> a) For each t, $\langle M^n \rangle_t \xrightarrow{d} v_t$;
>
> b) There are constants $c_n \downarrow 0$ such that
>
> $$\lim_{n \to \infty} P \left\{ \sup_{t \geq 1} |\Delta M^n_t| \leq c_n \right\} = 1.$$
>
> Then there exists a continuous Gaussian martingale M, with $\langle M \rangle_t = v_t$ for all t, such that $M^n \xrightarrow{d} M$ on the function space $D[0, 1]$.

If we choose $e = c_n = 1/n$ (and let $\text{Var}\{q(0)\} = 0$), then

$$v_t = 2kT \frac{g(V)}{V} t.$$

For a fixed voltage V, the rates $r_T(V)$ and $f_T(V)$ are constant, and the quadratic (or predictable) variation [44, p. 417] is

$$\langle M^e \rangle_t = e^2 \left[f_T(V) + r_T(V) \right] t.$$

Thus, applying the theorem to our case, the sequence of martingales M^e converges to a Gaussian martingale M over $[0, 1]$ (and hence, presumably, over any finite interval). Now, since M is a continuous zero-mean martingale with variance growing linearly with time, it is necessarily a scaled Brownian motion by a theorem of Doob [45, Thm. 11.9].

3.7.2 The Forward Equation

The goal of this section will be to show that, for a circuit consisting of a linear capacitor driven by a noisy nonlinear device, the forward equation for the Poisson process description converges to the Fokker-Planck equation for a Gaussian description as the size of the Poisson jumps grows smaller.

Consider the forward equation

$$
\begin{aligned}
\frac{\partial \rho(t, q)}{\partial t} &= f_T(v_\alpha(q + e)) \, \rho(t, q + e) + r_T(v_\beta(q - e)) \, \rho(t, q - e) \\
&\quad - f_T(v_\alpha(q)) \, \rho(t, q) - r_T(v_\beta(q)) \, \rho(t, q).
\end{aligned}
\tag{3.82}
$$

As $e \to 0$, we would like to see this become some sort of Fokker-Planck equation. The idea is to use the definition of derivative to replace the expressions on the right-hand side.

The centered approximation for the first derivative is

$$
\frac{dh}{dx} = \lim_{e \to 0} \frac{h(x + e) - h(x - e)}{2e},
$$

and the usual formula for the second derivative is

$$
\frac{d^2 h}{dx^2} = \lim_{h \to 0} \frac{h(x + e) - 2h(x) + h(x - e)}{e^2}.
$$

Because of the differences among v_α, v_β, and v, we need to evaluate the rates at the "average" voltage between the state we are leaving and the state we are going to. For a circuit with a single linear capacitor, the voltages in (3.82) are

$$
v_\alpha(q + e) = v_\beta(q) = \frac{q + e/2}{C}
\tag{3.83}
$$

$$
v_\alpha(q) = v_\beta(q - e) = \frac{q - e/2}{C}.
\tag{3.84}
$$

So, by Taylor expansion,

$$f_T\left(v_\alpha(q+e)\right) = f_T\left(\frac{q+e/2}{C}\right) = f_T\left(\frac{q+e}{C}\right) - \frac{e}{2C}f'_T\left(\frac{q+e}{C}\right).$$

We need to express all the derivatives in terms of d/dq:

$$\frac{df_T}{dq} = \frac{df_T}{dv}\frac{dv}{dq} = f'_T\frac{1}{C},$$

since $v = q/C$. Finally, we have

$$f_T\left(v_\alpha(q+e)\right) = f_T\left(\frac{q+e}{C}\right) - \frac{e}{2}\frac{df_T}{dq}, \tag{3.85}$$

and similarly for the other terms.

If we plug these first-order expansions into the forward equation (3.82), we arrive at

$$
\begin{aligned}
\frac{\partial\rho(t,q)}{\partial t} &= \left[f_T(v(q+e)) - \frac{e}{2}\frac{df_T(v(q+e))}{dq}\right]\rho(t,q+e)\\
&\quad + \left[r_T(v(q-e)) + \frac{e}{2}\frac{dr_T(q-e)}{dq}\right]\rho(t,q-e)\\
&\quad - \left[f_T(v(q)) + \frac{e}{2}\frac{df_T(v(q))}{dq}\right]\rho(t,q) - \left[r_T(v(q)) + \frac{e}{2}\frac{dr_T(v(q))}{dq}\right]\rho(t,q)\\[2mm]
&= f_T(v(q+e))\,\rho(t,q+e) + r_T(v(q-e))\,\rho(t,q-e)\\
&\quad - \left[f_T(v(q)) + r_T(v(q))\right]\rho(t,q)\\
&\quad + \frac{e}{2}\left[-\left.\frac{df_T}{dq}\right|_{q+e}\rho(t,q+e) + \left.\frac{df_T}{dq}\right|_q\rho(t,q)\right.\\
&\qquad\quad \left. + \left.\frac{dr_T}{dq}\right|_{q-e}\rho(t,q-e) - \left.\frac{dr_T}{dq}\right|_q\rho(t,q)\right].
\end{aligned}
$$

Now, using the centered derivative expressions, we obtain

$$
\begin{aligned}
\frac{\partial \rho(t,q)}{\partial t} &= \frac{\partial}{\partial q}\left[e\Big(f_T(v(q)) - r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[e^2\Big(f_T(v(q)) + r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad + \frac{e}{2}\left[-\left.\frac{df_T}{dq}\right|_{q+e}\rho(t,q+e) + \left.\frac{df_T}{dq}\right|_{q}\rho(t,q)\right. \\
&\qquad\qquad \left. + \left.\frac{dr_T}{dq}\right|_{q-e}\rho(t,q-e) - \left.\frac{dr_T}{dq}\right|_{q}\rho(t,q)\right].
\end{aligned}
$$

The terms in the second line are non-centered derivative formulas

$$
\left.\frac{df_T}{dq}\right|_{q+e}\rho(t,q+e) - \left.\frac{df_T}{dq}\right|_{q}\rho(t,q) = \frac{\partial}{\partial q}\left(\frac{df_T}{dq}\rho(t,q)\right)
$$

(and similarly for dr_T/dq). It is not clear why these can be applied in this case; perhaps further Taylor expansions of $f_T(v_\alpha(\cdot))$ would allow use of centered formulas. In any case, by substituting these in, we get

$$
\begin{aligned}
\frac{\partial \rho(t,q)}{\partial t} &= \frac{\partial}{\partial q}\left[e\Big(f_T(v(q)) - r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[e^2\Big(f_T(v(q)) + r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad + \frac{e^2}{2}\left[-\frac{\partial}{\partial q}\left(\frac{df_T}{dq}\rho(t,q)\right) - \frac{\partial}{\partial q}\left(\frac{dr_T}{dq}\rho(t,q)\right)\right] \\[2mm]
&= \frac{\partial}{\partial q}\left[e\Big(f_T(v(q)) - r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[e^2\Big(f_T(v(q)) + r_T(v(q))\Big)\rho(t,q)\right] \\
&\quad - \frac{1}{2}\frac{\partial}{\partial q}\left(\frac{d[e^2(f_T + r_T)]}{dq}\rho(t,q)\right).
\end{aligned}
$$

Now, we are ready to apply (3.72) and (3.76).

$$
\begin{aligned}
\frac{\partial \rho(t,q)}{\partial t} &= \frac{\partial}{\partial q}\left[g(v(q))\,\rho(t,q)\right] + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[2kT\,\frac{g(v(q))}{v(q)}\rho(t,q)\right] \\
&\quad - \frac{1}{2}\frac{\partial}{\partial q}\left[\frac{\partial}{\partial q}\left(2kT\frac{g(v(q))}{v(q)}\right)\,\rho(t,q)\right]
\end{aligned}
$$

This is the $(\zeta = 1)$-FPE.

3.7.3   Summary of Convergence Issues

In the limit that the electron charge goes to zero, the Poisson noise model no longer depends on the load. The rates of the Poisson counters depend on the average voltage (3.84) before and after a jump of one electron, but these voltages are the same if the electron charge is infinitesimal. This limit of the model yields the same circuit equations as the nonlinear Gaussian model for $\zeta = 1$. Unfortunately, Section 3.6.2 showed that, even for $\zeta = 1$, the Gaussian model still cannot be considered as a useful model for nonlinear devices because, when the device is biased at a fixed current, the model predicts an noise amplitude that is incompatible with the prediction when the device is biased at a fixed voltage. It is impossible to test the prediction of the Poisson model at a constant current, because this would create a cutset of current sources, in violation of the rules of circuit theory.

3.8   Final Thoughts

This chapter has presented a great deal of mathematics. It is hoped that the reader obtained some insight into stochastic differential equations and the Fokker-Planck equation. If not, at least the difficult derivations are now out of the way, and we may proceed to use the results in the following chapters.

Chapter 4

A Lossless Multiport Driven by the Shot-Noise Model and Nyquist-Johnson Resistors

4.1 Introduction

This chapter generalizes the tests of Chapter 2 for circuits driven by noise. The template for these generalization is [9], which considered the Nyquist-Johnson noise model driving a general lossless network. This chapter will feature both the Nyquist-Johnson Gaussian model and the shot-noise Poisson model in the same circuit, driving a multiport inductor and a multiport capacitor.

Other extensions are found in [46, 47]. These papers present specific examples such as a diode driving a time-varying capacitor; a diode driving a parallel inductor-capacitor combination; and a parallel circuit consisting of a resistor and a diode driving a capacitor and an inductor. These examples are special cases of the formulation of this chapter. The reader who finds the mathematics in this chapter too daunting or dry may prefer these specific examples.
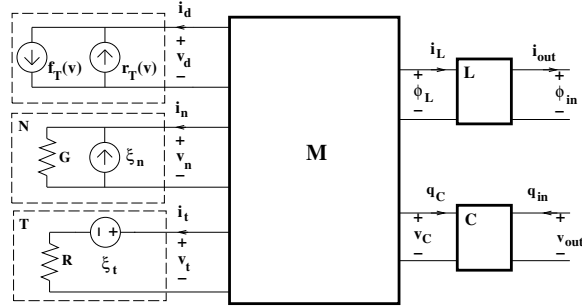
Figure 4-1: Three noise models driving a multiport capacitor and a multiport inductor through a linear, lossless, memoryless interconnection box.

The time-varying capacitor in [46] is used to show that work may be extracted from a heat engine consisting of two nonlinear resistors. A direct result of the increasing entropy for this circuit is that the efficiency of this heat engine is bounded by the Carnot efficiency. This argument is not reproduced here.

The circuit for consideration is shown in Fig. 4-1, where $\mathbf{M}$ is a linear, lossless, and memoryless interconnection box that may contain transformers or gyrators. It connects the three types of noisy devices to a multiport inductor and a multiport capacitor. The resistor box N consists of Norton-form resistors, and the box T consists of Thévenin-form.

The following equation describes the box:

$$
\begin{bmatrix} \mathbf{i}_C \\ \mathbf{v}_L \\ \mathbf{v}_d \\ \mathbf{v}_n \\ \mathbf{i}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} & \mathbf{J} & \mathbf{K} \\ -\mathbf{B}^T & \mathbf{F} & \mathbf{H} & \mathbf{P} & \mathbf{Q} \\ -\mathbf{D}^T & -\mathbf{H}^T & \mathbf{0} & \mathbf{S} & \mathbf{U} \\ -\mathbf{J}^T & -\mathbf{P}^T & -\mathbf{S}^T & \mathbf{W} & \mathbf{X} \\ -\mathbf{K}^T & -\mathbf{Q}^T & -\mathbf{U}^T & -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \\ \mathbf{i}_d \\ \mathbf{i}_n \\ \mathbf{v}_t \end{bmatrix} \tag{4.1}
$$

The entire matrix must be antisymmetric because the box is lossless, giving rise to the structure and constraining $\mathbf{A}$, $\mathbf{F}$, $\mathbf{W}$, and $\mathbf{Y}$ to be antisymmetric, as well. The

(3,3) entry need only be antisymmetric for a lossless box; however, we also need to rule out the delta-functions of current from one shot-noise device being "gyrated" into voltages for other devices. The (voltage-dependent) rates are only defined for finite voltages. The other matrices on the diagonal can be non-zero, however, for multiports containing transformers and gyrators.

If the multiport is reciprocal, then many of the submatrices must be zero: $\mathbf{A}$, $\mathbf{F}$, $\mathbf{H}$, $\mathbf{K}$, $\mathbf{P}$, $\mathbf{S}$, $\mathbf{W}$, and $\mathbf{Y}$. There are no gyrators to convert voltages to currents or vice-versa. In this case, the matrix equation is

$$
\begin{bmatrix} \mathbf{i}_C \\ \mathbf{v}_L \\ \mathbf{v}_d \\ \mathbf{v}_n \\ \mathbf{i}_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B} & \mathbf{D} & \mathbf{J} & \mathbf{0} \\ -\mathbf{B}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q} \\ -\mathbf{D}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} \\ -\mathbf{J}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{Q}^T & -\mathbf{U}^T & -\mathbf{X}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \\ \mathbf{i}_d \\ \mathbf{i}_n \\ \mathbf{v}_t \end{bmatrix}
$$

The matrix formulation of Eq. (4.1) rules out some circuits. In particular, it forbids non-controllable circuits, such as those containing capacitor loops or inductor cutsets. Further, an inductor may not be in series with a shot-noise device, because this would constrain their currents to be equal, rather than independent variables. A specific example is the series diode-LC circuit. In such a circuit, all three elements have the same current flowing through them, so there is only *one* independent current variable, whereas the equations show two ($\mathbf{i}_L$ and $\mathbf{i}_d$). A similar problem would occur with a capacitor voltage and a "gyrated" diode current.

We choose as independent variables $\mathbf{i}_d$, $\mathbf{i}_n$, and $\mathbf{v}_t$, because these variables contain stochastic terms. The diode currents are random impulses of current. The Norton-form resistors in box N have current noise sources, whereas the Thévenin-form resistors T have voltage noise sources.

Since $\mathbf{i}_L$ and $\mathbf{v}_C$ are independent variables, they do not have delta-functions

"forced" through them by the multiport interconnection; hence it is safe to differentiate these quantities to generate the dynamics of the circuit ($i_C = C \, dv_C/dt$ and $v_L = L \, di_L/dt$ for linear devices). It is fortuitous that the formulation rules out this circuit, because differentiating the delta-functions of current from the diodes would produce doublets in the inductor voltages or capacitor currents.

Hence, our choice of representation seems to have ruled out all of the pathologies that we need to rule out for simple interconnections.

In principle, one does not need two forms for the simple linear resistor, since they are indistinguishable at their ports. However, once we have chosen the independent variables $\mathbf{i}_n$ and $\mathbf{v}_t$, it is most convenient for the matrix representation (4.1) to use the Thévenin form for resistors that are connected to voltage-controlled ports and the Norton form for those connected to current-controlled ports. Consider the following two examples. In Fig. 4-2 (a), a Thévenin form resistor is connected to a capacitor. KVL constrains $v_t = v_C$, but this resistor model has v_t as the independent variable and we have chosen v_C as an independent state variable. Hence, for an RC loop, it is easier to use the Norton model. In Fig. 4-2 (b), a Norton form resistor is connected to an inductor. KCL constrains $i_n = i_L$, but this resistor model has i_n as the independent variable and we have chosen i_L as an independent state variable. Hence, for an RL loop, we prefer the Thévenin model. The solution presented in this paper assumes that the Norton model is used for resistors that drive current-controlled ports and the Thévenin model for resistors that drive voltage-controlled ports of the interconnection box $\mathbf{M}$. The solution in [9] does not properly point out that for the circuit of Fig. 4-2 (b), a further step is necessary to convert the noise current into a voltage.

4.2   The Forward Equation

The matrix representation from the previous section is a stochastic differential equation (SDE). In the most general case, the capacitors and inductors may be nonlinear,

(a)   (b)
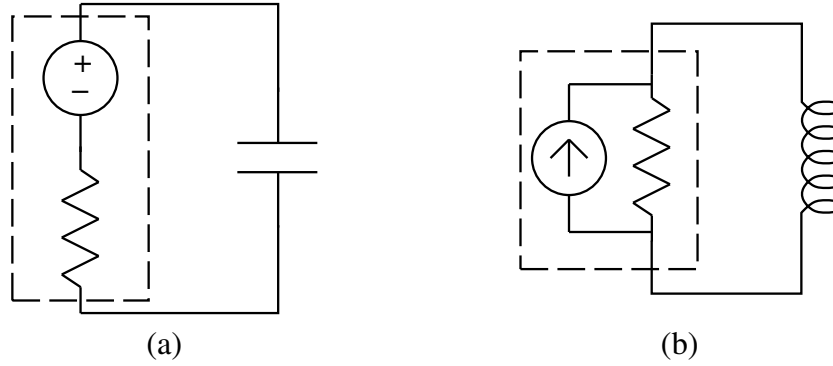
Figure 4-2: Linear resistors in inconvenient circuits

in which case it is best to choose the capacitor charge $\mathbf{q}$ and inductor flux ϕ as state variables.

$$\mathbf{i}_C = \frac{d\mathbf{q}}{dt} \qquad \text{and} \qquad \mathbf{v}_L = \frac{d\phi}{dt}$$

Using these relations in the matrix equation (4.1), the corresponding state evolution equation is

$$d \begin{bmatrix} \mathbf{q} \\ \phi \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} dt + \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} \mathbf{i}_d\, dt + \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{i}_n \\ \mathbf{v}_t \end{bmatrix} dt. \tag{4.2}$$

Using the differential form (3.33) for stochastic differential equations driven by Poisson counters, the shot-noise model for the diodes states

$$\mathbf{i}_d = e\left\{ d\mathbf{N}_f\left(\mathbf{v}_d\right) - d\mathbf{N}_r\left(\mathbf{v}_d\right) \right\}, \tag{4.3}$$

where $\mathbf{N}_f$ is a vector of forward currents (each of which only depends on its own voltage for our diode model, but the notation is already very cumbersome), and $\mathbf{N}_f$ is a vector of reverse currents. The voltage $\mathbf{v}_d$ is given by the third row of the matrix

equation (4.1),

$$\mathbf{v}_d = - \begin{bmatrix} \mathbf{D}^T & \mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} + \begin{bmatrix} \mathbf{S} & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{i}_n \\ \mathbf{v}_t \end{bmatrix}. \tag{4.4}$$

For the resistors, using the reference directions in Fig. 4-1, we have

$$\mathbf{i}_n = \mathbf{G}\,\mathbf{v}_n - \xi_n \tag{4.5}$$

$$\mathbf{v}_t = \mathbf{R}\,\mathbf{i}_t + \xi_t, \tag{4.6}$$

where $\mathbf{G}$ is a diagonal matrix of the conductances of the Norton-form resistors, $\mathbf{R}$ is the diagonal matrix of the resistances of the Thévenin-form resistors, and the ξ terms are vectors of Gaussian white-noise sources with the appropriate power spectral densities. Then, using the last two rows of the matrix equation (4.1), we can solve for

$$\begin{bmatrix} \mathbf{i}_n \\ \mathbf{v}_t \end{bmatrix} = \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1}$$
$$\times \left\{ \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \left(- \begin{bmatrix} \mathbf{J} & \mathbf{P} \\ \mathbf{K} & \mathbf{Q} \end{bmatrix}^T \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} + \begin{bmatrix} -\mathbf{S}^T \\ -\mathbf{U}^T \end{bmatrix} \mathbf{i}_d \right) + \begin{bmatrix} -\xi_n \\ \xi_t \end{bmatrix} \right\} \tag{4.7}$$

Unfortunately, it does not seem possible to invert both sets of equations for the diodes and the resistors. Ideally, we need to rewrite (4.3) and (4.7) so that they depend only on the state variables $\mathbf{v}_C$ and $\mathbf{i}_L$. But we are stuck with $\mathbf{i}_n$ and $\mathbf{v}_t$ inside the Poisson counter arguments.

A greater concern is guaranteeing that the diode voltages are always finite. At any instant that a Poisson counter fires, causing a delta-function of current in some diode, this current could (depending on $\mathbf{S}^T$ and $\mathbf{U}^T$) potentially cause a delta-function of current $\mathbf{i}_n$ or voltage $\mathbf{v}_t$ in (4.7), which could (depending on $\mathbf{S}$ and $\mathbf{U}$) then feed back

into the diode voltage in (4.4). (In fact, by writing out a few equations, one sees that for any i, j such that $S_{jk} \neq 0$, diode current i_{d_j} will cause delta-functions in i_{n_k} and therefore v_{d_j}; similarly for $U_{jk} \neq 0$.) We do not allow singular voltages for the forward and reverse rates of the Poisson counters, so we must constrain $\mathbf{S} = \mathbf{0}$ and $\mathbf{U} = \mathbf{0}$. This is sufficient to allow us to proceed.

One further remark, though, at this point: it would make some physical sense to insist that every diode have a capacitor connected across its terminals, corresponding to the junction capacitance of the diode. The capacitor would integrate the current out of the diodes and ensure a finite (if discontinuous) voltage. However, we need not make this restriction, corresponding to $\mathbf{H} = \mathbf{0}$, and our Poisson model might in fact be used for some nonlinear device other than a diode. The $\mathbf{H}$ matrix gyrates an inductor current into a diode voltage; the capacitor voltages and inductor currents are the "smoothest" quantities in the circuit, so it is physically sensible to require that the diode voltage depend only on some linear combination of these quantities.

Under the constraints $\mathbf{S} = \mathbf{0}$ and $\mathbf{U} = \mathbf{0}$, we can return to the stochastic differential equation (SDE) formulation (4.2)

$$
\begin{aligned}
d \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\phi} \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} dt \\
&+ \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} e \left\{ d\mathbf{N}_f \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} \right) \right. \\
&\qquad\qquad \left. - d\mathbf{N}_r \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} \right) \right\} \\
&+ \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \\
&\quad \times \left\{ \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \left(\begin{bmatrix} -\mathbf{J}^T & -\mathbf{P}^T \\ -\mathbf{K}^T & -\mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} \right) \right. dt \qquad \text{(continued)}
\end{aligned}
$$

$$
+ \begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix} \begin{bmatrix} -\mathbf{dw}_n \\ \mathbf{dw}_t \end{bmatrix} \Bigg\}
$$

$$
= \left\{ \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} + \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \right.
$$

$$
\times \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} -\mathbf{J}^T & -\mathbf{P}^T \\ -\mathbf{K}^T & -\mathbf{Q}^T \end{bmatrix} \left. \vphantom{\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}} \right\} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} dt
$$

$$
+ \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} e \left\{ d\mathbf{N}_f \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} \right) \right.
$$

$$
\left. -d\mathbf{N}_r \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{i}_L \end{bmatrix} \right) \right\}
$$

$$
+ \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1}
$$

$$
\times \begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix} \begin{bmatrix} -\mathbf{dw}_n \\ \mathbf{dw}_t \end{bmatrix}, \tag{4.8}
$$

where by $\sqrt{2kT\mathbf{G}}$ we mean $\sqrt{2kT}\,\mathbf{G}^{1/2}$, and the matrix square-root is well-defined, because $\mathbf{G}$ and $\mathbf{R}$ are positive diagonal matrices.

In the nonlinear case, the capacitor voltage and inductor current may be expressed in terms of the stored energies as

$$
\mathbf{v}_C \;\; = \nabla_{\mathbf{q}} E_C(\mathbf{q}, \mathbf{q}_{in}) \;\; \triangleq \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \tag{4.9}
$$

$$
\mathbf{i}_L \;\; = \nabla_{\boldsymbol{\phi}} E_L(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \;\; \triangleq \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}), \tag{4.10}
$$

where $\boldsymbol{\phi}_{in}$ and $\mathbf{q}_{in}$ are inputs to the system. Using these definitions,

$$
d \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\phi} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} + \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \right.
$$
$$
\times \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} -\mathbf{J}^T & -\mathbf{P}^T \\ -\mathbf{K}^T & -\mathbf{Q}^T \end{bmatrix} \left. \right\} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \end{bmatrix} dt
$$
$$
+ \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} e \left\{ d\mathbf{N}_f \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \end{bmatrix} \right) \right.
$$
$$
\left. - d\mathbf{N}_r \left(\begin{bmatrix} -\mathbf{D}^T & -\mathbf{H}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \end{bmatrix} \right) \right\}
$$
$$
+ \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1}
$$
$$
\times \begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix} \begin{bmatrix} -d\mathbf{w}_n \\ d\mathbf{w}_t \end{bmatrix} \tag{4.11}
$$

For a stochastic differential equation of the form

$$
d\mathbf{x} = \mathbf{a}(\mathbf{x}) \, dt + \mathbf{b} \, d\mathbf{w} + \sum_{i=1}^m \mathbf{g}_i \, dN_i(\lambda_i(\mathbf{x})),
$$

where $\mathbf{a}$ is a vector, $\mathbf{b}$ is a matrix, $d\mathbf{w}$ is a vector of unit-variance Gaussian white noises, $\mathbf{g}_i$ are constant vectors, and N_i are Poisson counters with (state-dependent) rates λ_i, it was shown in Section 3.5 that the forward equation is given by Eq. (3.61), namely,

$$
\frac{\partial \rho(\mathbf{x}, t)}{\partial t} = -\nabla^T \left[\mathbf{a}(\mathbf{x}) \, \rho(t, \mathbf{x}) \right] + \frac{1}{2} \sum_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\mathbf{b} \mathbf{b}^T \right)_{ij} \rho(t, \mathbf{x})
$$
$$
+ \sum_{i=1}^m \lambda_i(\mathbf{x} - \mathbf{g}_i) \rho(\mathbf{x} - \mathbf{g}_i, t) - \sum_{i=1}^m \lambda_i(\mathbf{x}) \rho(\mathbf{x}, t)
$$

where $\nabla^T = [\partial/\partial x_1, \partial/\partial x_2, \ldots]$ is a vector of partial derivatives with respect to all the variables and $\left(\mathbf{bb}^T\right)_{ij}$ is the (i,j)-th element of the matrix. It will be useful for the remainder of this chapter to write a vector form for the term containing $\mathbf{bb}^T$; note that this matrix does not depend on $\mathbf{x}$.

$$\sum_{i,j} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left(\mathbf{bb}^T\right)_{ij} \rho(t,\mathbf{x}) = \sum_{i,j} \left(\mathbf{bb}^T\right)_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \rho(t,\mathbf{x}) = \nabla^T \mathbf{bb}^T \nabla \rho(t,\mathbf{x})$$

where both ∇^T and ∇ are understood to operate on $\rho(t,\mathbf{x})$. Because of the linear algebra identity $\mathbf{x}^T \mathbf{M} \mathbf{y} = \mathrm{tr}\left\{\mathbf{M}\mathbf{y}\mathbf{x}^T\right\}$, where $\mathrm{tr}\{\}$ is the trace operator, this can also be expressed as

$$\nabla^T \mathbf{bb}^T \nabla \rho(t,\mathbf{x}) = \mathrm{tr}\left\{\mathbf{bb}^T \nabla \nabla^T \rho(t,\mathbf{x})\right\}$$

For the case of Eq. (4.11),

$$\mathbf{a}(\mathbf{q},\boldsymbol{\phi}) = \left\{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} + \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1} \right.$$
$$\left. \times \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} -\mathbf{J}^T & -\mathbf{P}^T \\ -\mathbf{K}^T & -\mathbf{Q}^T \end{bmatrix}\right\} \begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix} \qquad (4.12)$$

and

$$\mathbf{b} = \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1} \begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix} \qquad (4.13)$$

(Note: "elements" of these matrices are, in general, matrices as well; for example, the (1,1) element of $\mathbf{b}$ has the same dimensions as $\mathbf{J}$.) Also, the constant vectors in front of the Poisson counters are

$$\mathbf{g}_i = \pm \begin{bmatrix} \mathbf{D} \\ \mathbf{H} \end{bmatrix} e$$

where the sign depends on whether the particular counter N_i is a forward or reverse

counter. Therefore, the forward equation for Eq. (4.11) is

$$
\begin{aligned}
\frac{d}{dt}\, &\rho(t, \mathbf{q}, \boldsymbol{\phi}) \\
&= -\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right] \left[\mathbf{a}(\mathbf{q}, \boldsymbol{\phi})\, \rho(t, \mathbf{q}, \boldsymbol{\phi})\right] \\
&\quad + \frac{1}{2} \left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right] (\mathbf{b}\mathbf{b}^T) \begin{bmatrix} \nabla_{\mathbf{q}} \\ \nabla_{\boldsymbol{\phi}}^T \end{bmatrix} \rho(t, \mathbf{q}, \boldsymbol{\phi}) \\
&\quad + \sum_{i=1}^{n_d} \lambda_{f_i}(\tilde{v}_i^+)\, \rho(\mathbf{q} - e\, \mathbf{D}_i, \boldsymbol{\phi} - e\, \mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{f_i}(v_i^+)\, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \\
&\quad + \sum_{i=1}^{n_d} \lambda_{r_i}(\tilde{v}_i^-)\, \rho(\mathbf{q} + e\, \mathbf{D}_i, \boldsymbol{\phi} + e\, \mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-)\, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \quad (4.14)
\end{aligned}
$$

In this equation, λ_{f_i} is the rate of the i-th forward Poisson process (similarly for λ_{r_i}),

$$
\frac{dN_{f_i}(v_i)}{dt} = \lambda_{f_i}(v_i^+) \qquad \frac{dN_{r_i}(v_i)}{dt} = \lambda_{r_i}(v_i^-) \qquad (4.15)
$$

and these rates depend on the "effective" voltages, defined as

$$
\begin{aligned}
v_i^+ \quad &\triangleq \quad \text{the effective voltage on the i-th diode when the state} \\
&\qquad \text{jumps from } (\mathbf{q}, \boldsymbol{\phi}) \text{ to } (\mathbf{q} + e\mathbf{D}_i, \boldsymbol{\phi} + e\mathbf{H}_i) \qquad\qquad (4.16) \\
v_i^- \quad &\triangleq \quad \text{the effective voltage on the i-th diode when the state} \\
&\qquad \text{jumps from } (\mathbf{q}, \boldsymbol{\phi}) \text{ to } (\mathbf{q} - e\mathbf{D}_i, \boldsymbol{\phi} - e\mathbf{H}_i) \qquad\qquad (4.17) \\
\tilde{v}_i^+ \quad &\triangleq \quad \text{the voltage for a jump from } (\tilde{\mathbf{q}} - e\mathbf{D}_i, \tilde{\boldsymbol{\phi}} - e\mathbf{H}_i) \text{ to } (\tilde{\mathbf{q}}, \tilde{\boldsymbol{\phi}}) \qquad (4.18) \\
\tilde{v}_i^- \quad &\triangleq \quad \text{the voltage for a jump from } (\tilde{\mathbf{q}} + e\mathbf{D}_i, \tilde{\boldsymbol{\phi}} + e\mathbf{H}_i) \text{ to } (\tilde{\mathbf{q}}, \tilde{\boldsymbol{\phi}}) \qquad (4.19)
\end{aligned}
$$

The exact form of these voltages will be given later.

4.3   Equilibrium Density

Given the forward equation (4.14), the first question to be answered is: what is the equilibrium density? Thermodynamic Requirement #2 states it must be the Gibbs density:

$$\rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = A \exp\left[-E_{LC}(\mathbf{q}, \boldsymbol{\phi})/kT\right], \tag{4.20}$$

where E_{LC} is the energy stored in the capacitors and inductors, and A serves to normalize the equation [10, 11, 23].

Ref. [9] does not explicitly show this for the general linear resistor case, and the matrices here are considerably different from that paper, so it should be verified explicitly that this distribution is an equilibrium. The two drift terms of the first line of (4.14) tend to concentrate the density at the origin. This is balanced by the diffusion terms on the second line. The four jump terms also affect the distribution, but whether they concentrate or spread it is determined by the rates λ_{f_i} and λ_{r_i}.

4.3.1   Drift Terms

The drift terms of the forward equation (4.14) are

$$-\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right] \left[\mathbf{a}(\mathbf{q}, \boldsymbol{\phi})\, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})\right], \tag{4.21}$$

where $\mathbf{a}$ is defined in Eq. (4.12) and is quite complicated. The first term,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix},$$

which describes noiseless transfer of energy between the inductors and capacitors, will vanish because of its antisymmetry and the reciprocity of the inductor and capacitor

(required by losslessness).

$$\begin{bmatrix} \nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \\ \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \end{bmatrix}$$

$$= \nabla_{\mathbf{q}}^T \mathbf{A} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \nabla_{\mathbf{q}}^T \mathbf{B} \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})$$
$$- \nabla_{\boldsymbol{\phi}}^T \mathbf{B}^T \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \nabla_{\boldsymbol{\phi}}^T \mathbf{F} \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})$$

$$= \nabla_{\mathbf{q}}^T \mathbf{A} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \mathbf{B} \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \nabla_{\mathbf{q}}^T \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})$$
$$- \mathbf{B}^T \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \nabla_{\boldsymbol{\phi}}^T \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \nabla_{\boldsymbol{\phi}}^T \mathbf{F} \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})$$

$$= \nabla_{\mathbf{q}}^T \mathbf{A} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \nabla_{\boldsymbol{\phi}}^T \mathbf{F} \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}), \qquad (4.22)$$

where the last equality follows because

$$\nabla_{\mathbf{q}} \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \left(\frac{-\nabla_q E_{LC}}{kT} \right) = \frac{-\mathbf{f}(\mathbf{q}, \mathbf{q}_{in})}{kT} \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \qquad (4.23)$$

$$\nabla_{\boldsymbol{\phi}} \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \left(\frac{-\nabla_\phi E_{LC}}{kT} \right) = \frac{-\mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in})}{kT} \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}), \qquad (4.24)$$

by the definitions of the storage element constitutive relations, Eqs. (4.9) and (4.10).

Of course, if $\mathbf{A} = \mathbf{0}$ and $\mathbf{F} = \mathbf{0}$ (which corresponds to there being no gyrators in the interconnection box), then the two remaining terms vanish. However, in general, $\mathbf{A}$ and $\mathbf{F}$ are only assumed antisymmetric. Writing out terms of the second inner product of Eq. (4.22),

$$\frac{\partial}{\partial \phi_i} F_{ij} \, g_j(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) + \frac{\partial}{\partial \phi_j} F_{ji} \, g_i(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = 0, \quad \text{for } i \neq j. \quad (4.25)$$

By antisymmetry, $F_{ij} = -F_{ji}$ (and hence $F_{ii} = 0$). Further, F_{ij} is a constant and has

no effect on the derivative, so we can factor it out.

$$
\begin{aligned}
0 \;\overset{?}{=}\;\; & \frac{\partial}{\partial \phi_i}\left[g_j(\boldsymbol{\phi},\boldsymbol{\phi}_{in})\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi})\right] - \frac{\partial}{\partial \phi_j}\left[g_i(\boldsymbol{\phi},\boldsymbol{\phi}_{in})\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi})\right] \\
=\;\; & \frac{\partial g_j(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_i}\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) + g_j(\boldsymbol{\phi},\boldsymbol{\phi}_{in})\,\frac{\partial \rho_{eq}(\mathbf{q},\boldsymbol{\phi})}{\partial \phi_i} \\
& - \left[\frac{\partial g_i(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_j}\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) + g_i(\boldsymbol{\phi},\boldsymbol{\phi}_{in})\,\frac{\partial \rho_{eq}(\mathbf{q},\boldsymbol{\phi})}{\partial \phi_j}\right] \\
=\;\; & \left[\frac{\partial g_j(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_i} - \frac{\partial g_i(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_j}\right]\rho_{eq}(\mathbf{q},\boldsymbol{\phi}),
\end{aligned}
\tag{4.26}
$$

where the last equality follows by application of (4.24) expressed component-wise. The factor

$$
\left[\frac{\partial g_j(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_i} - \frac{\partial g_i(\boldsymbol{\phi},\boldsymbol{\phi}_{in})}{\partial \phi_j}\right]
\tag{4.27}
$$

tests reciprocity of the multiport inductor, and a lossless inductor must in fact be reciprocal. Hence, this factor is zero, and we have verified (4.26).

Similarly,

$$
\frac{\partial}{\partial q_i}A_{ij}\,f_j(\mathbf{q},\mathbf{q}_{in})\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) + \frac{\partial}{\partial q_j}A_{ji}\,f_i(\mathbf{q},\mathbf{q}_{in})\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) = 0, \quad \text{for } i \neq j, \tag{4.28}
$$

because the multiport capacitor is also lossless and reciprocal.

The remaining term of Eq. (4.21), a complicated product, is not so simple to analyze. In particular, it is unclear how to calculate the inverse, since the submatrices need not have the same dimensions. Let us define

$$
\begin{aligned}
\tilde{\mathbf{a}} \;\overset{\triangle}{=}\; & -\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \\
& \times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix}
\end{aligned}
\tag{4.29}
$$

Then the non-zero drift terms are

$$
-\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right]\left[\tilde{\mathbf{a}}(\mathbf{q}, \boldsymbol{\phi})\, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi})\right]
$$

$$
= \left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right]
\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}
\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}
\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}
\begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \end{bmatrix}
\rho_{eq}(\mathbf{q}, \boldsymbol{\phi}). \tag{4.30}
$$

Some similar factors appear in the diffusion terms.

4.3.2 Diffusion Terms

The diffusion terms of the forward equation (4.14) are

$$
\frac{1}{2}\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right](\mathbf{b}\mathbf{b}^T)
\begin{bmatrix} \nabla_{\mathbf{q}} \\ \nabla_{\boldsymbol{\phi}}^T \end{bmatrix}
\rho_{eq}(\mathbf{q}, \boldsymbol{\phi}), \tag{4.31}
$$

where $\mathbf{b}$ is defined in Eq. (4.13). Let us compute the matrix $(\mathbf{b}\mathbf{b}^T)$.

$$
(\mathbf{b}\mathbf{b}^T) = \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}
\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}
\begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \sqrt{2kT\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \sqrt{2kT\mathbf{R}} \end{bmatrix}
\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-T}
\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}^T
$$

$$
= 2kT \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}
\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}
\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-T}
\begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \tag{4.32}
$$

Recalling Eqs. (4.23) and (4.24) for the partial derivatives of the equilibrium density,

$$
\frac{1}{2} \begin{bmatrix} \nabla_\mathbf{q}^T, \nabla_\phi^T \end{bmatrix} (\mathbf{b}\mathbf{b}^T) \begin{bmatrix} \nabla_\mathbf{q} \\ \nabla_\phi^T \end{bmatrix} \rho_{eq}(\mathbf{q}, \phi)
$$

$$
= \begin{bmatrix} \nabla_\mathbf{q}^T, \nabla_\phi^T \end{bmatrix} \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} -\mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ -\mathbf{h}(\phi, \phi_{in}) \end{bmatrix} \rho_{eq}(\mathbf{q}, \phi).
$$

$$(4.33)$$

4.3.3 Combining Drift and Diffusion Terms

Note the similarities in the first several factors (in fact, the whole first line on the right-hand side) of Eqs. (4.30) and (4.33). Combining these equations and collecting the common factors (but being careful with minus signs), the drift and diffusion terms of the forward equation are

$$
-\begin{bmatrix} \nabla_\mathbf{q}^T, \nabla_\phi^T \end{bmatrix} \begin{bmatrix} \mathbf{a}(\mathbf{q}, \phi) \, \rho_{eq}(\mathbf{q}, \phi) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \nabla_\mathbf{q}^T, \nabla_\phi^T \end{bmatrix} (\mathbf{b}\mathbf{b}^T) \begin{bmatrix} \nabla_\mathbf{q} \\ \nabla_\phi^T \end{bmatrix} \rho_{eq}(\mathbf{q}, \phi)
$$

$$
= \begin{bmatrix} \nabla_\mathbf{q}^T, \nabla_\phi^T \end{bmatrix} \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \left\{ \mathbf{I} - \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \right\}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\phi, \phi_{in}) \end{bmatrix} \rho_{eq}(\mathbf{q}, \phi).
$$

$$(4.34)$$

Let us consider the third line. Following a trick in [9], a creative factorization manipulates Eq. (4.34) into a very convenient form.

$$
\left\{ I - \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-T} \right\}
$$

$$
= \left\{ \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^T - I \right\} \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-T}
$$

$$
= \left\{ I - \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix}^T \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} - I \right\} \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-T}
$$

$$
= \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix}^T \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-T},
$$

where the last equality follows by the antisymmetry of W and Y.

It was shown above that

$$
\begin{bmatrix} \nabla_q & \nabla_\phi \end{bmatrix} \left\{ \begin{bmatrix} A & B \\ -B^T & F \end{bmatrix} \begin{bmatrix} f(q) \\ h(\phi) \end{bmatrix} \rho_{eq}(t, q, \phi) \right\} = 0, \tag{4.35}
$$

because the matrices A and F were antisymmetric. Hence, for Eq. (4.34), we would like to show that

$$
\begin{bmatrix} J & K \\ P & Q \end{bmatrix} \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-1} \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}
$$

$$
\times \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix}^T \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \left(I - \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} W & X \\ -X^T & Y \end{bmatrix} \right)^{-T} \begin{bmatrix} J^T & P^T \\ K^T & Q^T \end{bmatrix}
$$

is antisymmetric. Following the creative factorization, this is automatic, because the

matrix

$$\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}^T$$

is antisymmetric, and for any antisymmetric matrix $\mathbf{A}$, $\mathbf{BAB}^T$ is also antisymmetric. (Of course, $\mathbf{G}$ and $\mathbf{R}$ are diagonal, and hence equal to their transposes.) Recall that the inner product terms of Eq. (4.35) involving the (1,2) and (2,1) submatrices will cancel because the differentiation of ρ_{eq} will bring down $\mathbf{f}$ or $\mathbf{h}$. Further, the (1,1) and (2,2) submatrices will cancel by reciprocity of the multiport inductor and capacitor:

$$\frac{\partial f_j(\mathbf{q})}{\partial q_i} = \frac{\partial f_i(\mathbf{q})}{\partial q_j} \qquad \text{and} \qquad \frac{\partial g_j(\boldsymbol{\phi})}{\partial \phi_i} = \frac{\partial g_i(\boldsymbol{\phi})}{\partial \phi_j}. \tag{4.36}$$

Therefore, the drift and diffusion terms for the Gibbs distribution sum to zero.

4.3.4   Jump Terms

The jump terms of the forward equation (4.14) are

$$+ \sum_{i=1}^{n_d} \lambda_{f_i}(\tilde{v}_i^+) \, \rho_{eq}(\mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{f_i}(v_i^+) \, \rho_{eq}(t, \mathbf{q}, \boldsymbol{\phi})$$
$$+ \sum_{i=1}^{n_d} \lambda_{r_i}(\tilde{v}_i^-) \, \rho_{eq}(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \, \rho_{eq}(t, \mathbf{q}, \boldsymbol{\phi}). \tag{4.37}$$

These remaining four terms of (4.14) must cancel. It is certainly sufficient if they cancel for each i; we claim that it is necessary. For example, if there is only one diode, its terms cannot cancel against those for other diodes. We expect to show that

$$\lambda_{r_i}(v_i^-) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = \lambda_{f_i}(\tilde{v}_i^+) \, \rho_{eq}(\mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i), \quad \text{for all } i \tag{4.38}$$

$$\lambda_{f_i}(v_i^+) \, \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) = \lambda_{r_i}(\tilde{v}_i^-) \, \rho_{eq}(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i), \quad \text{for all } i. \tag{4.39}$$

Note that, by the definitions of the voltages in Eqs. (4.16) and (4.19), that v_i^+ and $\tilde{v}_i^-$ both relate to jumps between $(\mathbf{q}, \boldsymbol{\phi})$ and $(\mathbf{q} + e\mathbf{D}_i, \boldsymbol{\phi} + e\mathbf{H}_i)$, in the forward direction for v_i^+ and the reverse direction for $\tilde{v}_i^-$.

It is also necessary to find expressions for the "shifted densities,"

$$\rho_{eq}(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) \quad \text{and} \quad \rho_{eq}(\mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i).$$

By reversing the definitions of $\mathbf{f}$ in (4.9) and $\mathbf{h}$ in (4.10), the energy may be expressed as

$$E_{LC}(\mathbf{q}, \boldsymbol{\phi}) = \oint_\gamma \mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}}, \tag{4.40}$$

where γ is a curve that starts at $(\mathbf{0}, \mathbf{0})$ and ends at $(\mathbf{q}, \boldsymbol{\phi})$. Note that this integral defining the energy E_{LC} is path-independent because the inductor and capacitor are reciprocal. Then, by using a simple property of the exponential,

$$\begin{aligned}
\rho_{eq}&(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) \\
&= \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \cdot \exp\left[\frac{-1}{kT}E_{LC}(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) + \frac{1}{kT}E_{LC}(\mathbf{q}, \boldsymbol{\phi})\right] \\
&= \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \cdot \exp\left[\frac{-1}{kT}\oint_\alpha \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}}\right)\right], \tag{4.41}
\end{aligned}$$

where α is a curve (any curve, by path-independence) that goes from $(\mathbf{q}, \boldsymbol{\phi})$ to $(\mathbf{q} + e\mathbf{D}_i, \boldsymbol{\phi} + e\mathbf{H}_i)$. Similarly,

$$\rho_{eq}(\mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) = \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \cdot \exp\left[\frac{-1}{kT}\oint_\beta \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}}\right)\right], \tag{4.42}$$

where β goes from $(\mathbf{q}, \boldsymbol{\phi})$ to $(\mathbf{q} - e\mathbf{D}_i, \boldsymbol{\phi} - e\mathbf{H}_i)$.

Now, we will finally define the voltages at the instant of a jump as

$$v_i^+ = \tilde{v}_i^- = \frac{-1}{e}\oint_\alpha \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}}\right), \tag{4.43}$$

and similarly

$$v_i^- = \tilde{v}_i^+ = \frac{1}{e} \oint_\beta \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}} \right), \tag{4.44}$$

where α and β are defined as above. (The sign difference is intentional.) Let us consider the meaning of these integrals. First, the correspond to the voltages used in the definitions of r_n and f_n, Eqs. (2.30) and (2.31), in the single-capacitor case. Since the integral is path-independent, the curve α (and similarly for β) can simply be the straight line connecting its endpoints, yielding a parameterization of $\tilde{\mathbf{q}}$ and $\tilde{\boldsymbol{\phi}}$ as follows:

$$\begin{aligned}
\tilde{\mathbf{q}}(t) &= t(\mathbf{q} + e\,\mathbf{D}_i) + (1 - t)\mathbf{q} = \mathbf{q} + e\,\mathbf{D}_i\, t \\
d\tilde{\mathbf{q}} &= e\,\mathbf{D}_i\, dt
\end{aligned}$$

$$\begin{aligned}
\tilde{\boldsymbol{\phi}}(t) &= t(\boldsymbol{\phi} + e\,\mathbf{H}_i) + (1 - t)\boldsymbol{\phi} = \boldsymbol{\phi} + e\,\mathbf{H}_i\, t \\
d\tilde{\boldsymbol{\phi}} &= e\,\mathbf{H}_i\, dt,
\end{aligned}$$

where t runs from 0 to 1. The line integral can then be calculated explicitly,

$$\begin{aligned}
v_i^+ &= \frac{-1}{e} \oint_\alpha \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}} \right) \\
&= \frac{-1}{e} \int_0^1 \mathbf{f}^T \left(\mathbf{q} + e\,\mathbf{D}_i\, t \right) e\,\mathbf{D}_i\, dt + \frac{-1}{e} \int_0^1 \mathbf{h}^T \left(\boldsymbol{\phi} + e\,\mathbf{H}_i\, t \right) e\,\mathbf{H}_i\, dt \\
&= -\int_0^1 \mathbf{f}^T \left(\mathbf{q} + e\,\mathbf{D}_i\, t \right) \mathbf{D}_i\, dt - \int_0^1 \mathbf{h}^T \left(\boldsymbol{\phi} + e\,\mathbf{H}_i\, t \right) \mathbf{H}_i\, dt \\
&= -\int_0^1 \mathbf{D}_i^T \mathbf{f} \left(\mathbf{q} + e\,\mathbf{D}_i\, t \right)\, dt - \int_0^1 \mathbf{H}_i^T \mathbf{h} \left(\boldsymbol{\phi} + e\,\mathbf{H}_i\, t \right)\, dt, \tag{4.45}
\end{aligned}$$

where the last equality follows because the integrands are scalars, which are equal to their transposes. $\mathbf{D}_i^T$ is the transpose of the i-th column of $\mathbf{D}$, which makes it the i-th row of $\mathbf{D}^T$; similarly for $\mathbf{H}_i^T$. Since $\mathbf{D}_i^T$ and $\mathbf{H}_i^T$ are constant, they can be factored

out of the integral to express the voltages in vector form,

$$\mathbf{v}^+ = -\mathbf{D}^T \int_0^1 \mathbf{f}\left(\mathbf{q} + e\,\mathbf{D}_i\,t\right)\,dt - \mathbf{H}^T \int_0^1 \mathbf{h}\left(\boldsymbol{\phi} + e\,\mathbf{H}_i\,t\right)\,dt. \qquad (4.46)$$

Now compare this definition to (4.4): $\mathbf{v}^+$ is a vector of the effective voltage on each diode when the system makes a jump between the endpoints of the curve α, from $(\mathbf{q}, \boldsymbol{\phi})$ to $(\mathbf{q} + e\mathbf{D}_i, \boldsymbol{\phi} + e\mathbf{H}_i)$. It is in some sense an average voltage along the curve.

Now, considering $\mathbf{v}_i^-$, the parameterization of β is

$$
\begin{aligned}
\tilde{\mathbf{q}}(t) &= t(\mathbf{q} - e\,\mathbf{D}_i) + (1-t)\mathbf{q} = \mathbf{q} - e\,\mathbf{D}_i\,t \\
d\tilde{\mathbf{q}} &= -e\,\mathbf{D}_i\,dt
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\boldsymbol{\phi}}(t) &= t(\boldsymbol{\phi} - e\,\mathbf{H}_i) + (1-t)\boldsymbol{\phi} = \boldsymbol{\phi} - e\,\mathbf{H}_i\,t \\
d\tilde{\boldsymbol{\phi}} &= -e\,\mathbf{H}_i\,dt.
\end{aligned}
$$

The line integration happens exactly the same; the lack of a minus sign in the definition (4.44) is compensated by the minus signs in $d\tilde{\mathbf{q}}$ and $d\tilde{\boldsymbol{\phi}}$.

$$\mathbf{v}^- = -\mathbf{D}^T \int_0^1 \mathbf{f}\left(\mathbf{q} - e\,\mathbf{D}_i\,t\right)\,dt - \mathbf{H}^T \int_0^1 \mathbf{h}\left(\boldsymbol{\phi} - e\,\mathbf{H}_i\,t\right)\,dt \qquad (4.47)$$

The quantity $\mathbf{v}^+$ is a vector of the effective voltage on each diode when the system makes a jump between the endpoints of the curve β, from $(\mathbf{q}, \boldsymbol{\phi})$ to $(\mathbf{q} - e\mathbf{D}_i, \boldsymbol{\phi} - e\mathbf{H}_i)$.

Using the definition (4.43) in (4.41) and (4.44) in (4.42), we find

$$\rho_{eq}(\mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) = \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \cdot \exp\left[\frac{ev_i^+}{kT}\right] \qquad (4.48)$$

$$\rho_{eq}(\mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) = \rho_{eq}(\mathbf{q}, \boldsymbol{\phi}) \cdot \exp\left[\frac{-ev_i^-}{kT}\right], \qquad (4.49)$$

and then substituting the shifted densities into (4.38) and (4.39), we cancel the com-

mon factor $\rho_{eq}(\mathbf{q}, \boldsymbol{\phi})$ and conclude that

$$\frac{\lambda_{f_i}(v_i^+)}{\lambda_{r_i}(v_i^+)} = \exp\left[\frac{v_i^+}{v_T}\right] \tag{4.50}$$

and

$$\frac{\lambda_{r_i}(v_i^-)}{\lambda_{f_i}(v_i^-)} = \exp\left[\frac{-v_i^-}{v_T}\right] \tag{4.51}$$

where $v_T = kT/e$. These two conditions are really the same equation, since we can pick any $(\mathbf{q}, \boldsymbol{\phi})$ to evaluate v_i^+ and v_i^-. This condition is exactly the condition we derived in the single-capacitor case,

$$\frac{\lambda_{f_i}(v)}{\lambda_{r_i}(v)} = \exp\left[\frac{v}{v_T}\right], \quad \text{for all } v. \tag{4.52}$$

4.4   Increasing Entropy

Thermodynamic Requirement #3 states that entropy must be monotonically increasing in time. The entropy of the energy storage side is classically defined as [11, 21]

$$S_{LC} = -k \iint \rho \, \log \rho \, d\mathbf{q} \, d\boldsymbol{\phi}. \tag{4.53}$$

Only changes in entropy are physically significant; the entropy rate is

$$\begin{aligned}
\frac{dS_{LC}}{dt} &= -k \iint \frac{d\rho}{dt} \log \rho \, d\mathbf{q} \, d\boldsymbol{\phi} - k \iint \rho \frac{1}{\rho} \frac{d\rho}{dt} \, d\mathbf{q} \, d\boldsymbol{\phi} \\
&= -k \iint \frac{d\rho}{dt} \log \rho \, d\mathbf{q} \, d\boldsymbol{\phi}. \tag{4.54}
\end{aligned}$$

The second term from the product rule vanishes, because total probability is conserved.

The reservoir entropy rate is calculated by use of the First Law of Thermodynam-

ics, conservation of energy.

$$\frac{d\overline{E_{LC}}}{dt} = \frac{dW}{dt} - T\frac{dS_R}{dt}, \qquad (4.55)$$

where $\overline{E_{LC}}$ is the expected energy stored in the inductor and capacitor, W is the work done on the system, T is the reservoir temperature, and S_R is the entropy of the reservoir. The expected value of the energy is

$$\overline{E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})} = \iint E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})\, \rho(t, \mathbf{q}, \boldsymbol{\phi})\, d\mathbf{q}\, d\boldsymbol{\phi}. \qquad (4.56)$$

When calculating the time derivative of this quantity, E_{LC} depends on time through the arguments $\mathbf{q}_{in}$ and $\boldsymbol{\phi}_{in}$, but not through the dummy variables $\mathbf{q}$ or $\boldsymbol{\phi}$.

$$
\begin{aligned}
\frac{d\overline{E_{LC}}}{dt} &= \iint E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})\, \frac{d\rho(t, \mathbf{q}, \boldsymbol{\phi})}{dt}\, d\mathbf{q}\, d\boldsymbol{\phi} \\
&+ \iint \Bigg[\nabla_{q_{in}} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \cdot \frac{d\mathbf{q}_{in}}{dt} \\
&+ \nabla_{\phi_{in}} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \cdot \frac{d\boldsymbol{\phi}_{in}}{dt} \Bigg]\, \rho(t, \mathbf{q}, \boldsymbol{\phi})\, d\mathbf{q}\, d\boldsymbol{\phi}. \quad (4.57)
\end{aligned}
$$

The power into the system is

$$\frac{dW}{dt} = \nabla_{q_{in}} E_{LC} \cdot \frac{d\mathbf{q}_{in}}{dt} + \nabla_{\phi_{in}} E_{LC} \cdot \frac{d\boldsymbol{\phi}_{in}}{dt}, \qquad (4.58)$$

which follows from

$$\nabla_{q_{in}} E_{LC} = \mathbf{v}_{out} \qquad \frac{d\mathbf{q}_{in}}{dt} = \mathbf{i}_{in}$$

$$\nabla_{\phi_{in}} E_{LC} = \mathbf{i}_{out} \qquad \frac{d\boldsymbol{\phi}_{in}}{dt} = \mathbf{v}_{in},$$

according to [9], so that each term in (4.58) is a power, $v \cdot i$. Therefore, the resistor

entropy rate is

$$\frac{dS_R}{dt} = -\frac{1}{T} \iint E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \frac{d\rho(t, \mathbf{q}, \boldsymbol{\phi})}{dt} \, d\mathbf{q} \, d\boldsymbol{\phi}. \qquad (4.59)$$

The rate of change of the total entropy is

$$
\begin{aligned}
\frac{dS_{tot}}{dt} &= \frac{dS_{LC}}{dt} + \frac{dS_R}{dt} \\
&= -k \iint \frac{d\rho}{dt} \log \rho \, d\mathbf{q} \, d\boldsymbol{\phi} - \frac{1}{T} \iint E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \frac{d\rho}{dt} \, d\mathbf{q} \, d\boldsymbol{\phi} \\
&= -\iint \frac{d\rho(t, \mathbf{q}, \boldsymbol{\phi})}{dt} \left[k \log \rho + \frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \right] d\mathbf{q} \, d\boldsymbol{\phi}. \qquad (4.60)
\end{aligned}
$$

Note, of course, that at equilibrium, not only is $d\rho/dt = 0$, but also the term in square brackets reduces to the constant $k \log A$. Integrating any constant times $d\rho/dt$ over all space must yield zero by conservation of total probability.

Substituting the forward equation (4.14) for $d\rho/dt$ into (4.60) yields

$$
\begin{aligned}
\frac{dS_{tot}}{dt} = -\iint \Big\{ \ & -\left[\nabla_\mathbf{q}^T, \nabla_{\boldsymbol{\phi}}^T\right] \left[\mathbf{a}(\mathbf{q}, \boldsymbol{\phi}) \, \rho(t, \mathbf{q}, \boldsymbol{\phi})\right] \\
& + \frac{1}{2} \left[\nabla_\mathbf{q}^T, \nabla_{\boldsymbol{\phi}}^T\right] (\mathbf{b}\mathbf{b}^T) \begin{bmatrix} \nabla_\mathbf{q} \\ \nabla_{\boldsymbol{\phi}}^T \end{bmatrix} \rho(t, \mathbf{q}, \boldsymbol{\phi}) \\
& + \sum_{j=1}^{n_d} \lambda_{f_i}(\tilde{v}_i^+) \, \rho(\mathbf{q} - e \, \mathbf{D}_i, \boldsymbol{\phi} - e \, \mathbf{H}_i) - \sum_{j=1}^{n_d} \lambda_{f_i}(v_i^+) \, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \\
& + \sum_{j=1}^{n_d} \lambda_{r_i}(\tilde{v}_i^-) \, \rho(\mathbf{q} + e \, \mathbf{D}_i, \boldsymbol{\phi} + e \, \mathbf{H}_i) - \sum_{j=1}^{n_d} \lambda_{r_i}(v_i^-) \, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \Big\} \\
& \times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \right] d\mathbf{q} \, d\boldsymbol{\phi}. \qquad (4.61)
\end{aligned}
$$

In the first line, $\mathbf{a}$ can be replaced with $\tilde{\mathbf{a}}$, because the difference $\mathbf{a} - \tilde{\mathbf{a}}$ corresponds to noiseless drift in the LC subcircuit, away from equilibrium just as in the equilibrium situation. The first two lines will be attacked by use of integration by parts. The

third and fourth lines will be rearranged, using the ratio constraint (4.52).

The first two lines (multiplied by the last) are the entropy terms corresponding to the noise generated by the linear resistors. The fact that these terms give a non-negative contribution to the entropy rate was established in [9]. The third and fourth lines (again multiplied by the last) are the entropy from the shot-noise devices; this contribution is also nonnegative. Note that the two types of contributions do not interact: the density ρ and energy E_{LC} are functions of the state variables of the LC subcircuit, and do not "know" what sort of devices are providing the drive. Therefore, we may consider the entropy contributions separately. Separate the total entropy rate $\dot{S}_{tot}$ into the shot-noise or Poisson contribution, $\dot{S}_{tot,P}$, and the Nyquist-Johnson noise contribution, $\dot{S}_{tot,NJ}$.

$$\dot{S}_{tot} = \dot{S}_{tot,P} + \dot{S}_{tot,NJ}$$

The Poisson contribution terms are

$$
\begin{aligned}
\frac{dS_{tot,P}}{dt} = & -\iint \left\{ \sum_{i=1}^{n_d} \lambda_{f_i}(v_i^-)\, \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{f_i}(v_i^+)\, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right. \\
& \left. + \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^+)\, \rho(t, \mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-)\, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right\} \\
& \times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \right] d\mathbf{q}\, d\boldsymbol{\phi}, \qquad (4.62)
\end{aligned}
$$

where Eqs. (4.43) and (4.44) have replaced $\tilde{v}_i^-$ and $\tilde{v}_i^+$. For the next step, the relationship (4.52) between λ_{f_i} and λ_{r_i} is used to replace λ_{f_i}.

$$
\frac{dS_{tot,P}}{dt} = -\iint \left\{ \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \, \exp\left(v_i^-/v_T\right) \, \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) \right.
$$

$$
- \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^+) \, \exp\left(v_i^+/v_T\right) \, \rho(t, \mathbf{q}, \boldsymbol{\phi})
$$

$$
\left. + \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^+) \, \rho(t, \mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) - \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \, \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right\}
$$

$$
\times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \right] \, d\mathbf{q} \, d\boldsymbol{\phi} \qquad (4.63)
$$

It is helpful to collect terms with respect to λ_{r_i} at the two voltages, v_i^- and v_i^+.

$$
\frac{dS_{tot,P}}{dt} = -\iint \left\{ \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \left[\exp\left(v_i^-/v_T\right) \, \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right] \right.
$$

$$
\left. - \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^+) \left[\exp\left(v_i^+/v_T\right) \, \rho(t, \mathbf{q}, \boldsymbol{\phi}) - \rho(t, \mathbf{q} + e\,\mathbf{D}_i, \boldsymbol{\phi} + e\,\mathbf{H}_i) \right] \right\}
$$

$$
\times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) \right] \, d\mathbf{q} \, d\boldsymbol{\phi} \qquad (4.64)
$$

The first two lines differ only by a shift in the dummy variables of integration. Therefore, we can break up the integral, shift the dummy variables in the second line (and also in the third line), and then recombine. This is essentially the same procedure used in Section 2.5.3, where the summation was reindexed.

$$
\frac{dS_{tot,P}}{dt} = -\iint \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \left[\exp\left(v_i^-/v_T\right) \, \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right]
$$

$$
\times \left[\frac{1}{T} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) - \frac{1}{T} E_{LC}(\mathbf{q} - e\,\mathbf{D}_i, \mathbf{q}_{in}, \boldsymbol{\phi} - e\,\mathbf{H}_i, \boldsymbol{\phi}_{in}) \right.
$$

$$
\left. + k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) - k \log \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) \right] \, d\mathbf{q} \, d\boldsymbol{\phi} \quad (4.65)
$$

The difference in energies may be explicitly calculated,

$$E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in}) - E_{LC}(\mathbf{q} - e\,\mathbf{D}_i, \mathbf{q}_{in}, \boldsymbol{\phi} - e\,\mathbf{H}_i, \boldsymbol{\phi}_{in})$$
$$= -\oint_\beta \left(\mathbf{f}(\tilde{\mathbf{q}}) \cdot d\tilde{\mathbf{q}} + \mathbf{h}(\tilde{\boldsymbol{\phi}}) \cdot d\tilde{\boldsymbol{\phi}} \right) = -e\,v_i^-, \qquad (4.66)$$

where the line integral along β was calculated in Eq. (4.44). Using this result,

$$\frac{dS_{tot,P}}{dt} = -\iint \sum_{i=1}^{n_d} \lambda_{r_i}(v_i^-) \left[\exp\left(v_i^-/v_T\right) \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right]$$

$$\times \left[k\log\rho(t, \mathbf{q}, \boldsymbol{\phi}) - k\log\rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \frac{e v_i^-}{T} \right] d\mathbf{q}\,d\boldsymbol{\phi}$$

$$= -\iint \sum_{i=1}^{n_d} k\,\lambda_{r_i}(v_i^-) \left[\exp\left(v_i^-/v_T\right) \rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right]$$

$$\times \left[\log\rho(t, \mathbf{q}, \boldsymbol{\phi}) - \log\rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) - \frac{v_i^-}{v_T} \right] d\mathbf{q}\,d\boldsymbol{\phi}. \quad (4.67)$$

The terms in the square brackets are logarithmically related:

$$\frac{dS_{tot,P}}{dt} = -\iint \sum_{i=1}^{n_d} k\,\lambda_{r_i}(v_i^-) \left[ab - c\right] \times \left[\log c - \log ab\right]. \qquad (4.68)$$

Just as in Eq. (2.43), the product

$$[ab - c] \times [\log c - \log ab]$$

is always nonpositive because the logarithm is monotonically increasing. When combined with the minus sign in front of the integral, we find that

$$\frac{dS_{tot,P}}{dt} \geq 0, \qquad (4.69)$$

with equality only when $ab = c$, that is,

$$\exp\left(v_i^-/v_T\right)\,\rho(t, \mathbf{q} - e\,\mathbf{D}_i, \boldsymbol{\phi} - e\,\mathbf{H}_i) = \rho(t, \mathbf{q}, \boldsymbol{\phi}), \qquad \text{for all } i, \tag{4.70}$$

which is exactly the condition (4.49) for the equilibrium density ρ_{eq}.

Now, let us consider the Nyquist-Johnson contributions.

$$
\begin{aligned}
\frac{dS_{tot,NJ}}{dt} &= -\iint \left\{ -\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right] \left[\tilde{\mathbf{a}}(\mathbf{q}, \boldsymbol{\phi})\,\rho(t, \mathbf{q}, \boldsymbol{\phi})\right] \right. \\
&\qquad \left. + \frac{1}{2}\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right](\mathbf{bb}^T)\begin{bmatrix}\nabla_{\mathbf{q}} \\ \nabla_{\boldsymbol{\phi}}^T\end{bmatrix} \rho(t, \mathbf{q}, \boldsymbol{\phi}) \right\} \\
&\qquad \times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T}E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})\right]\, d\mathbf{q}\, d\boldsymbol{\phi} \\[2ex]
&= \iint \left\{ \left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right]\left(\tilde{\mathbf{a}}(\mathbf{q}, \boldsymbol{\phi})\,\rho(t, \mathbf{q}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{bb}^T)\begin{bmatrix}\nabla_{\mathbf{q}} \\ \nabla_{\boldsymbol{\phi}}^T\end{bmatrix} \rho(t, \mathbf{q}, \boldsymbol{\phi})\right) \right\} \\
&\qquad \times \left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T}E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})\right]\, d\mathbf{q}\, d\boldsymbol{\phi} \tag{4.71}
\end{aligned}
$$

Remember that because ρ must fall off exponentially fast such that its integral over all space is finite, the product term in integration by parts ("uv" in the formula $\int u\, dv = uv - \int v\, du$) always vanishes. This leaves

$$
\begin{aligned}
\frac{dS_{tot,NJ}}{dt} &= -\iint \left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right]\left[k \log \rho(t, \mathbf{q}, \boldsymbol{\phi}) + \frac{1}{T}E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \boldsymbol{\phi}, \boldsymbol{\phi}_{in})\right] \\
&\qquad \cdot \left(\tilde{\mathbf{a}}(\mathbf{q}, \boldsymbol{\phi})\,\rho(t, \mathbf{q}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{bb}^T)\begin{bmatrix}\nabla_{\mathbf{q}} \\ \nabla_{\boldsymbol{\phi}}^T\end{bmatrix} \rho(t, \mathbf{q}, \boldsymbol{\phi})\right)\, d\mathbf{q}\, d\boldsymbol{\phi}
\end{aligned}
$$

$$= -\iint \left(\frac{k}{\rho(t,\mathbf{q},\boldsymbol{\phi})} \left[\nabla_{\mathbf{q}}^T \rho(t,\mathbf{q},\boldsymbol{\phi}), \nabla_{\boldsymbol{\phi}}^T \rho(t,\mathbf{q},\boldsymbol{\phi}) \right] \right.$$

$$\left. + \frac{1}{T} \left[\mathbf{f}^T(\mathbf{q},\mathbf{q}_{in}), \mathbf{h}^T(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \right] \right)$$

$$\cdot \left(\tilde{\mathbf{a}}(\mathbf{q},\boldsymbol{\phi}) \, \rho(t,\mathbf{q},\boldsymbol{\phi}) - \frac{1}{2}(\mathbf{b}\mathbf{b}^T) \begin{bmatrix} \nabla_{\mathbf{q}} \, \rho(t,\mathbf{q},\boldsymbol{\phi}) \\ \nabla_{\boldsymbol{\phi}}^T \, \rho(t,\mathbf{q},\boldsymbol{\phi}) \end{bmatrix} \right) \, d\mathbf{q} \, d\boldsymbol{\phi}. \quad (4.72)$$

Recall the definitions of $\tilde{\mathbf{a}}$, Eq. (4.29), and $\mathbf{b}\mathbf{b}^T$, Eq. (4.32), from the equilibrium distribution test. Substituting these definitions into Eq. (4.72) and factoring out terms where possible, we find

$$\frac{dS_{tot,NJ}}{dt} =$$

$$\iint \left(\frac{k}{\rho(t,\mathbf{q},\boldsymbol{\phi})} \left[\nabla_{\mathbf{q}}^T \, \rho(t,\mathbf{q},\boldsymbol{\phi}), \nabla_{\boldsymbol{\phi}}^T \, \rho(t,\mathbf{q},\boldsymbol{\phi}) \right] + \frac{1}{T} \left[\mathbf{f}^T(\mathbf{q},\mathbf{q}_{in}), \mathbf{h}^T(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \right] \right)$$

$$\times \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix} \rho(t,\mathbf{q},\boldsymbol{\phi}) \right.$$

$$\left. + kT \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{q}} \, \rho(t,\mathbf{q},\boldsymbol{\phi}) \\ \nabla_{\boldsymbol{\phi}} \, \rho(t,\mathbf{q},\boldsymbol{\phi}) \end{bmatrix} \right\}$$

$$\times \, d\mathbf{q} \, d\boldsymbol{\phi}. \quad (4.73)$$

The extra factor in the next-to-last line for $\mathbf{b}\mathbf{b}^T$, compared with the middle line for $\tilde{\mathbf{a}}$, may be inserted because of a clever factorization trick from [9]. Note that for the symmetric form (in a simplified case for clarity),

$$\mathbf{x}\left(\mathbf{I} - \mathbf{G}\,\mathbf{W}\right)^{-1}\mathbf{G}\,\mathbf{x}^T$$

$$= \mathbf{x}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-1}\mathbf{x}^T$$

$$= \frac{1}{2}\mathbf{x}\left\{\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-1} + \left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-T}\right\}\mathbf{x}^T$$

$$= \frac{1}{2}\mathbf{x}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-1}\left\{\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{T} + \left(\mathbf{G}^{-1} - \mathbf{W}\right)\right\}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-T}\mathbf{x}^T$$

$$= \frac{1}{2}\mathbf{x}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-1}\left\{\mathbf{G}^{-1} - \mathbf{W}^T + \mathbf{G}^{-1} - \mathbf{W}\right\}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-T}\mathbf{x}^T$$

$$= \mathbf{x}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-1}\mathbf{G}^{-1}\left(\mathbf{G}^{-1} - \mathbf{W}\right)^{-T}\mathbf{x}^T$$

$$= \mathbf{x}\left(\mathbf{I} - \mathbf{G}\,\mathbf{W}\right)^{-1}\mathbf{G}\left(\mathbf{I} - \mathbf{G}\,\mathbf{W}\right)^{-1}\mathbf{x}^T.$$

Using this in the $\tilde{\mathbf{a}}$ term gives an expression in $\mathbf{b}\mathbf{b}^T$,

$$\left[\mathbf{f}^T(\mathbf{q},\mathbf{q}_{in}), \mathbf{h}^T(\phi,\phi_{in})\right]\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\phi,\phi_{in}) \end{bmatrix}\rho(t,\mathbf{q},\phi)$$

$$= \left[\mathbf{f}^T(\mathbf{q},\mathbf{q}_{in}), \mathbf{h}^T(\phi,\phi_{in})\right]\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}\left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-T}\begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\phi,\phi_{in}) \end{bmatrix}\rho(t,\mathbf{q},\phi)$$

$$= \frac{1}{2kT}\left[\mathbf{f}^T(\mathbf{q},\mathbf{q}_{in}), \mathbf{h}^T(\phi,\phi_{in})\right]\left(\mathbf{b}\mathbf{b}^T\right)\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\phi,\phi_{in}) \end{bmatrix}\rho(t,\mathbf{q},\phi). \tag{4.74}$$

This does not complete the analysis. Of the two terms on the first line of Eq. (4.73), the last expression only completes the second. The other term must also be re-expressed: first by integration by parts and then the factorization trick.

$$
\iint k \left[\nabla_{\mathbf{q}}^T \, \rho(t, \mathbf{q}, \phi), \nabla_{\phi}^T \, \rho(t, \mathbf{q}, \phi) \right]
$$

$$
\times \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\phi, \phi_{in}) \end{bmatrix} \, d\mathbf{q} \, d\phi
$$

$$
= -\iint k \, \rho(t, \mathbf{q}, \phi) \left[\nabla_{\mathbf{q}}^T, \nabla_{\phi}^T \right]
$$

$$
\times \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}(\mathbf{q}, \mathbf{q}_{in}) \\ \mathbf{h}(\phi, \phi_{in}) \end{bmatrix} \, d\mathbf{q} \, d\phi
$$

$$
= -\iint k \, \rho(t, \mathbf{q}, \phi) \left[\nabla_{\mathbf{q}}^T, \nabla_{\phi}^T \right] \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}
$$

$$
\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{q}} \\ \nabla_{\phi}^T \end{bmatrix} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \phi, \phi_{in}) \, d\mathbf{q} \, d\phi
$$

$$
= -\iint k \, \rho(t, \mathbf{q}, \phi) \left[\nabla_{\mathbf{q}}^T, \nabla_{\phi}^T \right] \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}
$$

$$
\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}
$$

$$
\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T}
$$

$$
\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{q}} \\ \nabla_{\phi}^T \end{bmatrix} E_{LC}(\mathbf{q}, \mathbf{q}_{in}, \phi, \phi_{in}) \, d\mathbf{q} \, d\phi
$$

$$= -\iint k\,\rho(t,\mathbf{q},\boldsymbol{\phi})\left[\nabla_{\mathbf{q}}^T, \nabla_{\boldsymbol{\phi}}^T\right]\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}$$

$$\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-T}$$

$$\times \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix}\,d\mathbf{q}\,d\boldsymbol{\phi}$$

$$= \iint k\,\left[\nabla_{\mathbf{q}}^T\,\rho(t,\mathbf{q},\boldsymbol{\phi}), \nabla_{\boldsymbol{\phi}}^T\,\rho(t,\mathbf{q},\boldsymbol{\phi})\right]\begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix}$$

$$\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-1}\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

$$\times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}\begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix}\right)^{-T}\begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix}\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix}\,d\mathbf{q}\,d\boldsymbol{\phi}$$

$$= \iint \frac{1}{2T}\left[\nabla_{\mathbf{q}}^T\,\rho(t,\mathbf{q},\boldsymbol{\phi}), \nabla_{\boldsymbol{\phi}}^T\,\rho(t,\mathbf{q},\boldsymbol{\phi})\right]\left(\mathbf{b}\mathbf{b}^T\right)\begin{bmatrix} \mathbf{f}(\mathbf{q},\mathbf{q}_{in}) \\ \mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in}) \end{bmatrix}\,d\mathbf{q}\,d\boldsymbol{\phi} \qquad (4.75)$$

Finally, we obtain (suppressing subscripts for ease of reading)

$$\frac{dS_{tot,NJ}}{dt} = \iint \frac{1}{2T\rho}\left[\frac{\mathbf{f}^T\rho}{kT} + \nabla_{\mathbf{q}}^T\,\rho, \frac{\mathbf{h}^T\rho}{kT} + \nabla_{\boldsymbol{\phi}}^T\,\rho\right]\left(\mathbf{b}\mathbf{b}^T\right)\begin{bmatrix} \mathbf{f}\rho + kT\,\nabla_{\mathbf{q}}\,\rho \\ \\ \mathbf{h}\rho + kT\,\nabla_{\boldsymbol{\phi}}\,\rho \end{bmatrix}\,d\mathbf{q}\,d\boldsymbol{\phi}.$$

$$(4.76)$$

Of course, at equilibrium,

$$\nabla_{\mathbf{q}}\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) = -\frac{1}{kT}\,\mathbf{f}(\mathbf{q},\mathbf{q}_{in})\,\rho_{eq} \quad \text{and} \quad \nabla_{\boldsymbol{\phi}}\,\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) = -\frac{1}{kT}\,\mathbf{h}(\boldsymbol{\phi},\boldsymbol{\phi}_{in})\,\rho_{eq},$$

so that $\frac{dS_{tot,NJ}}{dt} = 0$. Away from equilibrium, we have

$$
\begin{aligned}
\frac{dS_{tot,NJ}}{dt} = & \iint \frac{1}{2T\rho} \left[\frac{\mathbf{f}^T \rho}{kT} + \nabla_\mathbf{q}^T \rho, \frac{\mathbf{h}^T \rho}{kT} + \nabla_\phi^T \rho \right] \\
& 2kT \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \\
& \times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \\
& \times \begin{bmatrix} \mathbf{f}\rho + kT\,\nabla_\mathbf{q}\,\rho \\ \mathbf{h}\rho + kT\,\nabla_\phi\,\rho \end{bmatrix} d\mathbf{q}\,d\phi \\
= & \iint \frac{1}{T\rho} \left[\mathbf{f}^T\rho + kT\,\nabla_\mathbf{q}^T\,\rho, \mathbf{h}^T\rho + kT\,\nabla_\phi^T\,\rho \right] \begin{bmatrix} \mathbf{J} & \mathbf{K} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \\
& \times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \\
& \times \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \\
& \times \begin{bmatrix} \mathbf{f}\rho + kT\,\nabla_\mathbf{q}\,\rho \\ \mathbf{h}\rho + kT\,\nabla_\phi\,\rho \end{bmatrix} d\mathbf{q}\,d\phi \\
= & \iint \frac{1}{T\rho} \mathbf{x}^T \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{x}\, d\mathbf{q}\,d\phi, \hspace{3cm} (4.77)
\end{aligned}
$$

where

$$
\mathbf{x} = \left(\mathbf{I} - \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{bmatrix} \right)^{-T} \begin{bmatrix} \mathbf{J}^T & \mathbf{P}^T \\ \mathbf{K}^T & \mathbf{Q}^T \end{bmatrix} \begin{bmatrix} \mathbf{f}\rho + kT\,\nabla_\mathbf{q}\,\rho \\ \mathbf{h}\rho + kT\,\nabla_\phi\,\rho \end{bmatrix}.
$$

Since the resistances and conductances are assumed positive,

$$\frac{dS_{tot,NJ}}{dt} = \iint \frac{1}{T\rho} \mathbf{x}^T \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \mathbf{x} \, d\mathbf{q} \, d\boldsymbol{\phi} \geq 0. \tag{4.78}$$

Since both the Nyquist-Johnson (4.77) and the Poisson (4.67) entropy contributions are non-negative according to (4.78) and (4.69), respectively, and zero only at equilibrium, we have shown that

$$\frac{dS_{tot}}{dt} \geq 0, \tag{4.79}$$

with equality only at equilibrium.

Chapter 5

Heat Transfer between Noisy Devices

5.1 Introduction

This chapter will investigate the heat transferred between noisy devices. The question was motivated by [48], which asked: if one connected a diode and a linear resistor together in the same circuit, would there be any heat flow between them? If they are at the same temperature, of course, there should be no heat flow. However, the tests applied so far only ask about the total circuit behavior. This question is stated formally as follows:

Thermodynamic Requirement #4: No Heat Transfer between Two Devices at the Same Temperature

For any circuit consisting of two or more noisy devices, each in thermal contact with a thermal reservoir of a single temperature T, and any lossless lumped network, there should be no heat transfer between the devices, that is, no net power delivered or absorbed by any one of the devices. In contrast, heat should flow from the hotter to the

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cooler if the devices are in thermal contact with reservoirs at different temperatures, but the rate of flow will depend on specifics of the devices and the lossless network.

Ref. [49] showed a general result for power flow in each frequency band between noise sources. The specific result that two linear resistors with Gaussian noise models at different temperatures will exchange heat if and only if the temperatures are different, is independently derived in Section 5.2. Nyquist's original derivation of the linear resistor noise model was based in part on this assumption, but did not specify that the resistors had a Gaussian noise model.

In Section 5.3, random process arguments are used to find the power supplied by the two sources of a single diode model connected to a capacitor. The expression derived for the power is used in the remaining sections, which attempt to find the heat transferred between two diodes or between a diode and a resistor.

The average power supplied by a device is the expectation of the product of v and i through the device. Of course, the expectation is zero for a lossless device such as the capacitor. The "pure resistor" in the Norton-form model will dissipate power; therefore the parallel current noise source must, in expectation, supply power. For the diode, either shot noise source can supply or dissipate power, depending on the sign of the applied voltage. However, it turns out that, for this case, power must instead be calculated as a rate of change of energy.

Section 5.4 calculates the heat transferred between two nonlinear noisy devices, described by the Poisson shot-noise model, at different temperatures. Although we do not know the steady-state distribution for the charge on a capacitor driven by two Poisson devices, we can nonetheless show that heat is transferred from the warmer device to the cooler. There is no heat transfer when the temperatures are equal, even if the devices are different (differently-sized diodes, for example).

Section 5.5 calculates the heat transferred between a linear resistor, described by the Nyquist-Johnson model, and a nonlinear device, described by the Poisson model.
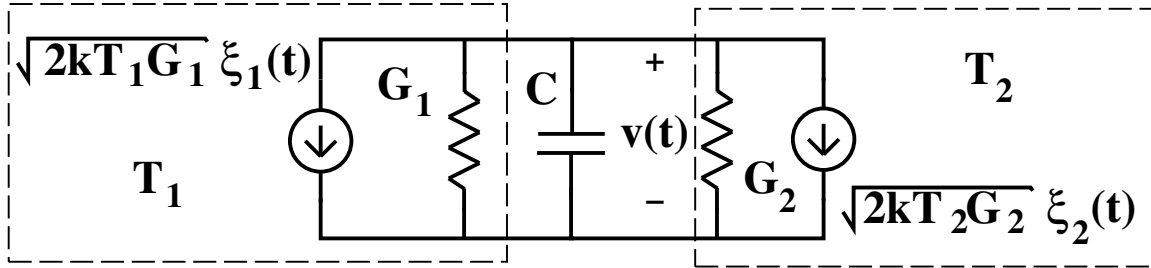
Figure 5-1: Two linear resistors at different temperatures driving a capacitor

In this case, the lack of an expression for the steady-state distribution prevents us from showing that heat is transferred when the devices are at different temperatures. However, when the temperatures are equal, the equilibrium distribution is known, and the devices do not transfer heat.

A related result is presented in [46], which constructs a heat engine from two nonlinear resistors at different temperatures and a time-varying capacitor switched between the two resistors. This is a nonlinear extension of the heat engine in [50].

After making the calculations in this chapter, it was found that Gunn [5] also considered the heat transfer between a linear resistor and a diode at different temperatures. However, that paper uses a linearized approximation to the diode.

5.2 Gaussian to Gaussian

The differential equation for the voltage in the circuit above is

$$C\frac{dv}{dt} = -v(t)\left(G_1 + G_2\right) - \sqrt{2kT_1G_1}\,\xi_1(t) - \sqrt{2kT_2G_2}\,\xi_2(t), \qquad (5.1)$$

where $\xi_1(t)$ and $\xi_2(t)$ are independent Gaussian random variables of unit variance. The Fokker-Planck equation is then

$$\frac{\partial \rho(t,v)}{\partial t} = \frac{\partial}{\partial v}\left[\frac{G_1 + G_2}{C}\, v\, \rho(t,v)\right] + \frac{kT_1G_1}{C^2}\frac{\partial^2 \rho(t,v)}{\partial v^2} + \frac{kT_2G_2}{C^2}\frac{\partial^2 \rho(t,v)}{\partial v^2}\ . \qquad (5.2)$$

Steady-state is achieved when

$$\frac{\partial \rho_{ss}(v)}{\partial t} = 0 = \frac{\partial}{\partial v}\left[\frac{G_1 + G_2}{C}\, v\, \rho_{ss}(v) + \frac{k\,(T_1G_1 + T_2G_2)}{C^2}\frac{\partial \rho_{ss}(v)}{\partial v}\right]$$

$$\frac{\partial \rho_{ss}(v)}{\partial v} = -\frac{G_1 + G_2}{k(T_1G_1 + T_2G_2)}\, v\, \rho_{ss}(v).$$

It is easy to see that the steady-state distribution will again be a (zero-mean) Gaussian in v. In fact, this was *a priori* known, since this is an LTI system with Gaussian inputs. The remaining free parameter is the variance. Defining

$$T_{\text{eff}} \triangleq \frac{T_1G_1 + T_2G_2}{G_1 + G_2}, \qquad (5.3)$$

the steady-state distribution is then

$$\rho_{ss}(v) = \frac{1}{\sqrt{2\pi kT_{\text{eff}}/C}}\exp\left[-\frac{C\,v^2}{2kT_{\text{eff}}}\right]. \qquad (5.4)$$

Note if $T_1 = T_2 = T$, then $T_{\text{eff}} = T$ and we recover the regular Gibbs distribution at a temperature T. If $G_1 = G_2 = G$, then $T_{\text{eff}} = (T_1 + T_2)/2$, the average of the temperatures. If the (electrical) conductances are not equal, then T_{eff} is a weighted average, corresponding to having different thermal conductivities.

Now let us consider the heat transfer when the two temperatures are different. Most thermodynamic calculations of this sort ask only for the power delivered by the complete model, pure conductor plus current source. However, it is quite simple to calculate the heat dissipated in each of the pure conductors, because this only

depends on the mean square voltage of the capacitor. Because this will be useful for later calculations, let us write it down.

$$P_{\text{diss},G_1} = G_1 \overline{v^2} = G_1 \frac{kT_{\text{eff}}}{C} \tag{5.5}$$

$$P_{\text{diss},G_2} = G_2 \overline{v^2} = G_2 \frac{kT_{\text{eff}}}{C} \tag{5.6}$$

If the conductors are equal, then the power dissipated in each conductor is equal. Any heat transfer must come because the two current sources provide different amounts of power to the circuit.

The power supplied by the current sources can be calculated using frequency-domain methods and current division. Because the circuit is linear, the power supplied by each source may be calculated separately and then added by superposition. The input current power spectral densities for the left source is

$$S_{ii,\text{in}} = 2kT_1 G_1.$$

The transfer function for the current division is

$$H_{12} = \frac{G_2}{G_1 + G_2 + j\omega C},$$

which gives the fraction of the input current that flows through the conductor G_2. The current power spectral density in conductor G_2 due to the left source is

$$S_{ii,\text{out}} = S_{ii,\text{in}} H_{12} H_{12}^* = 2kT_1 G_1 \frac{G_2^2}{(G_1 + G_2)^2 + \omega^2 C^2} . \tag{5.7}$$

The inverse Fourier transform of the power spectral density is the autocorrelation, $R_{ii,\text{out}}(t)$; the mean square current is the autocorrelation evaluated at $t = 0$. Thus,

the power supplied by source 1 and dissipated in conductor 2 is

$$P_{1\to 2} = \overline{i^2_{1\to 2}}/G_2 = \mathcal{F}^{-1}\left\{S_{ii,\text{out}}\right\}\big|_{t=0}/G_2 = \frac{kT_1 G_1 G_2}{C(G_1 + G_2)} \ . \tag{5.8}$$

By symmetry, the power supplied by source 1 and dissipated in conductor 1 is

$$P_{1\to 1} = \frac{kT_1 G_1^2}{C(G_1 + G_2)} \ . \tag{5.9}$$

Further,

$$P_{2\to 1} = \frac{kT_2 G_1 G_2}{C(G_1 + G_2)} \tag{5.10}$$

$$P_{2\to 2} = \frac{kT_2 G_2^2}{C(G_1 + G_2)} \ . \tag{5.11}$$

The total amount of power supplied to the circuit by both sources is

$$
\begin{aligned}
P_{\text{supp}} &= P_{1\to 2} + P_{1\to 1} + P_{2\to 1} + P_{2\to 2} \\
&= \frac{k}{C(G_1 + G_2)} \left[T_1\left(G_1 G_2 + G_1^2\right) T_2\left(G_1 G_2 + G_2^2\right)\right] \\
&= \frac{k}{C}\left[T_1 G_1 + T_2 G_2\right] .
\end{aligned}
\tag{5.12}
$$

The total power dissipated in the conductors is

$$
\begin{aligned}
P_{\text{diss}} &= P_{\text{diss},G_1} + P_{\text{diss},G_2} = (G_1 + G_2)\frac{kT_{\text{eff}}}{C} \\
&= \frac{k\left[T_1 G_1 + T_2 G_2\right]}{C} ,
\end{aligned}
\tag{5.13}
$$

where the last equality used the definition of T_{eff}, Eq. (5.3).

Thermodynamics asks about the power out of the complete Nyquist-Johnson

model. For the left-hand model in Fig. 5-1,

$$
\begin{aligned}
P_1 &= P_{1\to 2} + P_{1\to 1} - P_{\text{diss},G_1} \\
&= \frac{k\,T_1 G_1 G_2}{C(G_1 + G_2)} + \frac{k\,T_1 G_1^2}{C(G_1 + G_2)} - G_1 \frac{k\,T_{\text{eff}}}{C} \\
&= \frac{k\,G_1}{C}\left(\frac{T_1 G_2}{G_1 + G_2} + \frac{T_1 G_1}{G_1 + G_2} - \frac{T_1 G_1 + T_2 G_2}{G_1 + G_2} \right) \\
&= \frac{k\,G_1}{C}\left(\frac{T_1 G_2}{G_1 + G_2} - \frac{T_2 G_2}{G_1 + G_2} \right) \\
&= \frac{k\,G_1 G_2}{C(G_1 + G_2)}\,(T_1 - T_2)\,.
\end{aligned}
\tag{5.14}
$$

Similar calculations (or just conservation of energy) give the power out of the right-hand source:

$$
P_2 = \frac{k\,G_1 G_2}{C(G_1 + G_2)}\,(T_2 - T_1)\,.
\tag{5.15}
$$

For unequal temperatures, say $T_1 > T_2$, the left-hand source will supply power $P_1 > 0$; heat flows from the hotter conductor to the cooler. Of course, if the temperatures are equal, then both of these powers are zero.

5.3 Single-Device Poisson Model

In this section, the average power delivered by each of the two Poisson sources, forward and reverse, in the diode model will be calculated. Of course, the two sources together must supply no net power, because the capacitor is lossless. But it is not immediately clear how to calculate this power, since the current occurs in delta-functions. Calculating the expectation of $v \cdot i$ would require the distribution of voltage at firing times. It turns out that there is a simpler way to analyze the situation.

Gallager's treatment of Markov processes with countable state spaces is pertinent here. Although the state space in [13] is indexed by the non-negative integers, a doubly-infinite state space may be re-indexed such that our positive integers are
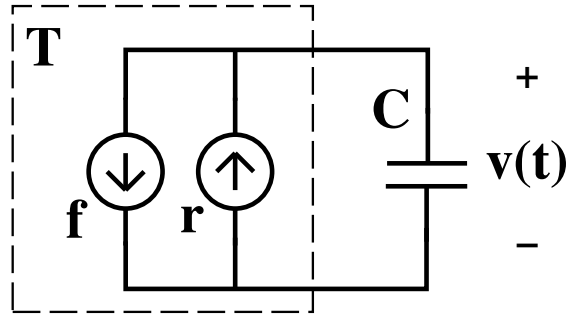
Figure 5-2: One diode driving a capacitor

assigned to odd integers and negative integers are assigned to even integers. In this case, the transitions of the diode model would be to "second neighbors" (plus the 0-1 transition), but the theory holds.

Gallager defines q_{ij} as the rate of transitions from i to j, and $\nu_i = \sum_j q_{ij}$ is the rate of transitions out of state i. Under certain conditions, the Markov process has a steady-state distribution $\{p_i\}$, where p_i is not only the steady-state probability of being in state i but also the time average fraction of time spent in state i. Therefore, the quantity

$$p_i \nu_i \frac{q_{ij}}{\nu_i} = p_i \, q_{ij}$$

is the time average rate of transitions of the whole Markov process along the particular arc from i to j. The quantity $p_i \nu_i$ is the steady-state rate at which transitions occur out of state i, and the fraction $\frac{q_{ij}}{\nu_i}$ gives the probability that the transition out is to state j (or the probability that the Poisson counter for the transition from i to j fires before any of the other counters out of state i).

The condition given in Gallager's Theorem 1 of Chapter 6 [13, p. 190] for existence and uniqueness of the steady-state probabilities $\{p_i\}$ is that

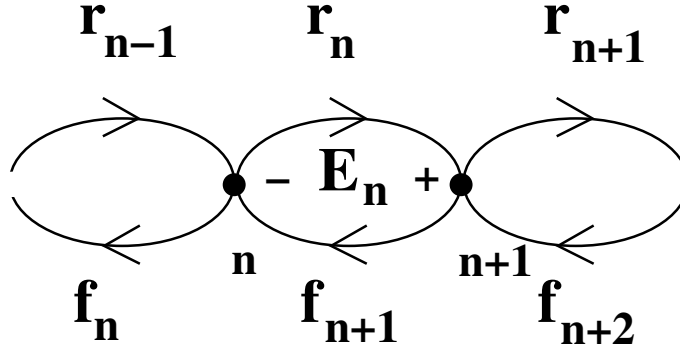$$\sum_i p_i \, \nu_i < \infty. \tag{5.16}$$

Figure 5-3: Markov process on a countable state space

For the diode model, the Gibbs equilibrium distribution is

$$p_n^o = A \ \exp\left(-\frac{n^2 e^2}{2CkT}\right),$$ (5.17)

where $1/A = \sum_n \exp\left(-\frac{n^2 e^2}{2CkT}\right)$ normalizes the distribution. This distribution exists, and Thermodynamic Requirement #2 states that it must be the equilibrium distribution. That this is the unique solution may be verified by checking Eq. (5.16) for the model:

$$
\begin{aligned}
\sum_i p_i \, \nu_i &= \sum_n p_n^o \left[f_n + r_n\right] \\
&= \sum_n A \ \exp\left(-\frac{n^2 e^2}{2CkT}\right) \frac{I_S}{e} \left[\exp\left(\frac{(n-1/2)e^2}{CkT}\right) + 1\right] \\
&= \sum_n A \ \frac{I_S}{e} \left[\exp\left(-\frac{(n^2 - 2n + 1)e^2}{2CkT}\right) + \exp\left(-\frac{n^2 e^2}{2CkT}\right)\right] \\
&= \sum_m A \ \frac{I_S}{e} \left[\exp\left(-\frac{m^2 e^2}{2CkT}\right)\right] + \sum_n A \ \frac{I_S}{e} \left[\exp\left(-\frac{n^2 e^2}{2CkT}\right)\right] \\
&= \frac{2I_S}{e} < \infty.
\end{aligned}
$$

Consider the Markov process shown in Fig. 5-3. E_n is the change in capacitor

energy when r_n fires, moving the process from n to $n + 1$ (hence $-E_n$ corresponds to the transition $n + 1$ to n). From the discussion above, $p_i\, q_{ij}$ is steady-state rate at which transitions occur from state i to j. For the capacitor charge chain,

$$p_n^o\, r_n = \text{steady-state rate at which transitions occur from state } n \text{ to } n + 1$$

$$p_n^o\, r_n\, E_n = \text{steady-state rate at which the energy } E_n \text{ is delivered by the reverse}$$
$$\text{source for transitions from } n \text{ to } n + 1$$

$$= \text{steady-state power delivered by the reverse source for transitions from}$$
$$n \text{ to } n + 1$$

$$\sum_n p_n^o\, r_n\, E_n = \text{steady-state power delivered by the reverse source on all transitions}$$

For the forward source, each transition out of state n changes the capacitor energy by $-E_{n-1}$, so that

$$-\sum_n p_n^o\, f_n\, E_{n-1} = \text{steady-state power delivered by the forward source on all}$$
$$\text{transitions}$$

For the diode model, the following equations were derived in Chapter 2.

$$p_n^o = A \exp\left(-\frac{n^2 e^2}{2CkT}\right) \tag{5.18}$$

$$r_n = \frac{I_S}{e} \tag{5.19}$$

$$f_n = \frac{I_S}{e} \exp\left(\frac{(n - 1/2)e^2}{CkT}\right) \tag{5.20}$$

$$E_n = \frac{(n+1)^2 e^2 - n^2 e^2}{2C} = \frac{(2n+1)e^2}{2C} \tag{5.21}$$

$$E_{n-1} = \frac{n^2 e^2 - (n-1)^2 e^2}{2C} = \frac{(2n-1)e^2}{2C} \tag{5.22}$$

Therefore,

$$\sum_n p_n^o\, r_n\, E_n = \sum_n A \exp\left(-\frac{n^2 e^2}{2CkT}\right) \frac{I_S}{e} \frac{(2n+1)e^2}{2C}$$

$$= \frac{I_S\,e}{2C}\, \sum_n A\,(2n+1)\exp\left(-\frac{n^2 e^2}{2CkT}\right)$$

$$= \frac{I_S\,e}{2C}, \tag{5.23}$$

where the last equality follows because the distribution is zero-mean $\left(\sum_n n p_n^o = 0\right)$ and normalized.

For the forward source,

$$
\begin{aligned}
-\sum_n p_n^o\,f_n\,E_{n-1} &= -\sum_n A\,\exp\left(-\frac{n^2 e^2}{2CkT}\right)\frac{I_S}{e}\exp\left(\frac{(n-1/2)e^2}{CkT}\right)\frac{(2n-1)e^2}{2C} \\
&= -\frac{I_S\,e}{2C}\,\sum_n A\,(2n-1)\exp\left(\frac{-n^2 e^2 + 2ne^2 - e^2}{2CkT}\right) \\
&= -\frac{I_S\,e}{2C}\,\sum_n A\,(2n-1)\exp\left(\frac{-(n-1)^2 e^2}{2CkT}\right) \\
&= -\frac{I_S\,e}{2C}\,\sum_m A\,(2m+1)\exp\left(\frac{-m^2 e^2}{2CkT}\right) \qquad (m = n - 1) \\
&= -\frac{I_S\,e}{2C}\,. \tag{5.24}
\end{aligned}
$$

So, as must have been the case, the power delivered by the reverse source is the opposite of that absorbed by the forward source.

What if the nonlinear device is not a diode? The net power out of the two Poisson sources can be shown to vanish by judicious use of the detailed balance criterion. Of course, this fact is not particularly interesting in this case, since it follows from the losslessness of the capacitor. Rewriting the summation for the forward source as

$$-\sum_n p_n^o\,f_n\,E_{n-1} = -\sum_n p_{n+1}^o\,f_{n+1}\,E_n,$$

it is clear that

$$\sum_n p_n^o\,r_n\,E_n - \sum_n p_{n+1}^o\,f_{n+1}\,E_n = \sum_n \left(p_n^o\,r_n - p_{n+1}^o\,f_{n+1}\right)\,E_n = 0, \tag{5.25}$$
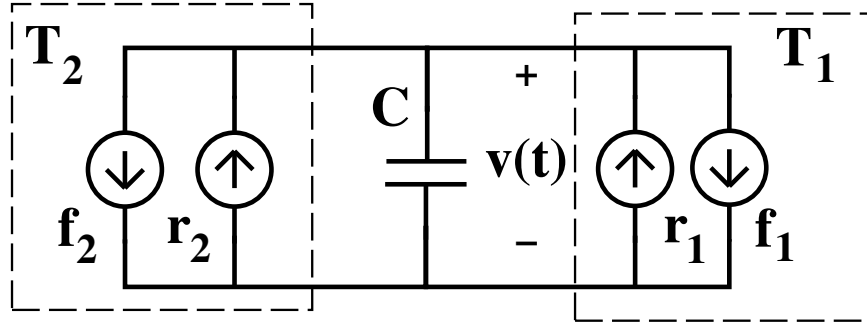
Figure 5-4: Two diodes at different temperatures driving a capacitor

because the term in parentheses is the detailed balance criterion. This trick will be more useful in the next section.

5.4 Poisson to Poisson

Given the result in Section 5.2 for two linear resistors, one might hope that the steady-state distribution for two diodes in a circuit would also be Gaussian. The forward equation for this circuit is

$$
\begin{aligned}
\frac{\partial \rho(t, q)}{\partial t} &= \left[f_1 \left(v_\alpha(q + e) \right) + f_2 \left(v_\alpha(q + e) \right) \right] \rho(t, q + e) \\
&\quad + \left[r_1 \left(v_\beta(q - e) \right) + r_2 \left(v_\beta(q - e) \right) \right] \rho(t, q - e) \\
&\quad - \left[f_1 \left(v_\alpha(q) \right) + f_2 \left(v_\alpha(q) \right) + r_1 \left(v_\beta(q) \right) + r_2 \left(v_\beta(q) \right) \right] \rho(t, q), \quad (5.26)
\end{aligned}
$$

where $q = ne$ is the number of positive charges on the top plate of the capacitor and $v_\alpha(q) = v_\beta(q - e) = (q - e/2)/C$ is the "effective" voltage for transitions from q to $q - e$ or vice-versa. The hypothesis is that there is a T_{eff} such that the steady-state distribution is given by

$$
\rho_{\text{hyp}}(q) \propto \exp \left[-\frac{q^2}{2CkT_{\text{eff}}} \right].
$$

Substituting this and the expressions for f and r for the diode into the forward equation (but dividing out the common constant factors), yields

$$
\begin{aligned}
0 \overset{?}{=}\ & \left[\exp\left(\frac{(n+1/2)e}{CkT_1/e}\right) + \exp\left(\frac{(n+1/2)e}{CkT_2/e}\right)\right]\exp\left(-\frac{(n+1)^2 e^2}{2CkT_{\text{eff}}}\right) \\
& +2\exp\left(-\frac{(n-1)^2 e^2}{2CkT_{\text{eff}}}\right) \\
& -\left[\exp\left(\frac{(n-1/2)e}{CkT_1/e}\right) + \exp\left(\frac{(n-1/2)e}{CkT_2/e}\right) + 2\right]\exp\left(-\frac{n^2 e^2}{2CkT_{\text{eff}}}\right) \quad (5.27)
\end{aligned}
$$

It does not seem possible to solve this equation for all n.

A second possibility for finding the equilibrium distribution is to use the detailed-balance criterion. Detailed balance, strictly speaking, is an equilibrium concept. However, it is still non-sensical for the probability to have a net flow in either direction. In fact, although the physical situation described here is not equilibrium, the steady-state distribution is an equilibrium of the Markov process.

The detailed-balance criterion for the Markov process corresponding to the two-device circuit is

$$
\rho_{ss}(q) r_{tot}(v_\beta(q)) = \rho_{ss}(q+e) f_{tot}(v_\alpha(q+e)),
$$

so, with two Poisson devices,

$$
\rho_{ss}(q)\Big[r_1(v_\beta(q)) + r_2(v_\beta(q))\Big] = \rho_{ss}(q+e)\Big[f_1(v_\alpha(q+e)) + f_2(v_\alpha(q+e))\Big]. \quad (5.28)
$$

Using the thermodynamic constraint derived in Chapter 2,

$$
\frac{f(v)}{r(v)} = \exp(v/v_T) \Leftrightarrow \frac{f_1(v_\alpha(q+e))}{r_1(v_\beta(q))} = \exp\left(\frac{qe + e^2/2}{CkT_1}\right), \quad (5.29)
$$

the ratio of adjacent states is expressed

$$
\frac{\rho_{ss}(q+e)}{\rho_{ss}(q)} = \frac{r_1(v_\beta(q)) + r_2(v_\beta(q))}{r_1(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_1}\right) + r_2(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_2}\right)}. \quad (5.30)
$$

As in Section 2.5, the probability of a charge $q = ne$ is then calculated by multiplying $(n-1)$ of these terms by the probability of $q = 0$, which is determined by normalization. Then, the power transferred by each source can be written in terms of these probabilities.

$$P_{r_1} = \sum_n \rho_{ss}(ne) r_1(v_\beta(ne)) E_n \tag{5.31}$$

$$P_{r_2} = \sum_n \rho_{ss}(ne) r_2(v_\beta(ne)) E_n \tag{5.32}$$

$$P_{f_1} = -\sum_n \rho_{ss}(ne) f_1(v_\alpha(ne)) E_{n-1} \tag{5.33}$$

$$P_{f_2} = -\sum_n \rho_{ss}(ne) f_2(v_\alpha(ne)) E_{n-1} \tag{5.34}$$

Note that if $T_1 = T_2$, then the exponentials in the denominator of Eq. (5.30) are equal, so that the fraction reduces to

$$\frac{\rho_{ss}(q+e)}{\rho_{ss}(q)} = \exp\left(-\frac{e v_\beta(q)}{kT}\right) = \exp\left(-\frac{qe + e^2/2}{CkT}\right), \tag{5.35}$$

which was calculated in Eq. (2.37). The steady-state distribution is the same as the equilibrium distribution used in Eq. (2.36),

$$\rho_{ss}(q) = A \exp\left(-\frac{q^2}{2CkT}\right),$$

and, in fact, this steady-state is also a physical equilibrium.

To verify equilibrium, it must be shown that no heat is transferred between the devices, even if they are not identical (in which case, the verification would be trivial by symmetry). Since the thermodynamic constraint (5.29) must hold for both devices, and since the equilibrium distribution has the ratio given by (5.35),

$$\begin{aligned}
\rho_{ss}(q+e) f_1(v_\alpha(q+e)) &= \exp\left(\frac{qe + e^2/2}{CkT}\right) \exp\left(-\frac{qe + e^2/2}{CkT}\right) \rho_{ss}(q) r_1(v_\beta(q)) \\
&= \rho_{ss}(q) r_1(v_\beta(q))
\end{aligned}$$

$$\rho_{ss}(q+e)f_2(v_\alpha(q+e)) = \rho_{ss}(q)r_2(v_\beta(q)).$$

Therefore, using the expressions derived above,

$$
\begin{aligned}
P_1 &= P_{r_1} + P_{f_1} \\
&= \sum_n \rho_{ss}(ne)r_1(v_\beta(ne))E_n - \sum_n \rho_{ss}(ne+e)f_1(v_\alpha(ne+e))E_n \\
&= \sum_n \Big[\rho_{ss}(ne)r_1(v_\beta(ne)) - \rho_{ss}(ne+e)f_1(v_\alpha(ne+e))\Big]E_n \\
&= 0,
\end{aligned}
$$

and also

$$
\begin{aligned}
P_2 &= P_{r_2} + P_{f_2} \\
&= \sum_n \rho_{ss}(ne)r_2(v_\beta(ne))E_n - \sum_n \rho_{ss}(ne+e)f_2(v_\alpha(ne+e))E_n \\
&= \sum_n \Big[\rho_{ss}(ne)r_2(v_\beta(ne)) - \rho_{ss}(ne+e)f_2(v_\alpha(ne+e))\Big]E_n \\
&= 0.
\end{aligned}
$$

Even without knowing the constitutive relation of the devices, we have shown that there is no heat transferred between the two – so long as they satisfy the thermodynamic constraint (5.29). The devices could be diodes with different saturation currents, or entirely different nonlinear devices. This result is actually not surprising, since ρ_{ss} for the two diodes is equal to ρ_{eq} for a single diode, so that the diodes cannot distinguish whether there is another diode connected to the same capacitor.

Suppose now that the temperatures are not equal. Let us pull an exponential out of the ratio of adjacent states, Eq. (5.30).

$$\frac{\rho_{ss}(q+e)}{\rho_{ss}(q)} = \frac{r_1(v_\beta(q)) + r_2(v_\beta(q))}{r_1(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_1}\right) + r_2(v_\beta(q))\exp\left(\frac{ev_\beta(q)}{kT_2}\right)}$$

$$= \exp\left(-\frac{ev_\beta(q)}{kT_1}\right) \frac{r_1(v_\beta(q)) + r_2(v_\beta(q))}{r_1(v_\beta(q)) + r_2(v_\beta(q))\exp\left[\frac{ev_\beta(q)}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]} \quad (5.36)$$

Also note that, since $v_\beta(q) = v_\alpha(q + e)$,

$$\frac{f_1(v_\alpha(ne + e))}{r_1(v_\beta(ne))} = \exp\left(\frac{ev_\beta(q)}{kT_1}\right).$$

When we calculate the product of the ratio of adjacent states and the ratio of forward and reverse rates for device 1, this exponential will cancel. In this case, the expression for the power supplied by device 1 can be simplified as follows.

$$\begin{aligned}
P_1 &= P_{r_1} + P_{f_1} \\
&= \sum_n \rho_{ss}(ne)r_1(v_\beta(ne))E_n - \sum_n \rho_{ss}(ne)f_1(v_\alpha(ne))E_{n-1} \\
&= \sum_n \rho_{ss}(ne)r_1(v_\beta(ne))E_n - \sum_n \rho_{ss}(ne + e)f_1(v_\alpha(ne + e))E_n \\
&= \sum_n \left[\rho_{ss}(ne)r_1(v_\beta(ne)) - \rho_{ss}(ne + e)f_1(v_\alpha(ne + e))\right]E_n \\
&= \sum_n \rho_{ss}(ne)\,r_1(v_\beta(ne))\,E_n\left[1 - \frac{\rho_{ss}(ne + e)}{\rho_{ss}(ne)}\frac{f_1(v_\alpha(ne + e))}{r_1(v_\beta(ne))}\right] \\
&= \sum_n \rho_{ss}(ne)\,r_1(v_\beta(ne))\,E_n \\
&\qquad \times \left[1 - \frac{r_1(v_\beta(ne)) + r_2(v_\beta(ne))}{r_1(v_\beta(ne)) + r_2(v_\beta(ne))\exp\left[\frac{ev_\beta(ne)}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]}\right] \\
&= \sum_n \rho_{ss}(ne)\,r_1(v_\beta(ne))\,\frac{(2n + 1)e^2}{2C} \\
&\qquad \times \left[1 - \frac{r_1(v_\beta(ne)) + r_2(v_\beta(ne))}{r_1(v_\beta(ne)) + r_2(v_\beta(ne))\exp\left[\frac{ev_\beta(q)}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]}\right] \\
&= \frac{e^2}{2C}\sum_n \rho_{ss}(ne)\,r_1(v_\beta(ne))\,(2n + 1) \\
&\qquad \times \left[1 - \frac{r_1(v_\beta(ne)) + r_2(v_\beta(ne))}{r_1(v_\beta(ne)) + r_2(v_\beta(ne))\exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]}\right] \quad (5.37)
\end{aligned}$$

Now suppose $T_1 > T_2$. For $n \geq 0$,

$$(2n+1) > 0 \quad \text{and} \quad \left(\frac{1}{T_2} - \frac{1}{T_1}\right) > 0 \quad \Rightarrow \quad \exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right] > 1$$

$$\Rightarrow \quad \frac{r_1(v_\beta(ne)) + r_2(v_\beta(ne))}{r_1(v_\beta(ne)) + r_2(v_\beta(ne))\exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]} < 1,$$

because the reverse rates $r_1(v_\beta(ne))$ and $r_2(v_\beta(ne))$ are always positive, and hence all the terms of the summation with $n \geq 0$ are positive. For $n < 0$,

$$(2n+1) < 0 \quad \text{and} \quad \left(\frac{1}{T_2} - \frac{1}{T_1}\right) > 0 \quad \Rightarrow \quad \exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right] < 1$$

$$\Rightarrow \quad \frac{r_1(v_\beta(ne)) + r_2(v_\beta(ne))}{r_1(v_\beta(ne)) + r_2(v_\beta(ne))\exp\left[\frac{e^2(n+1/2)}{Ck}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)\right]} > 1.$$

Now, both $(2n+1)$ and the term in the biggest square brackets are negative, so the product is again positive. Hence, the terms of the summation for all n are positive.

$$P_1 > 0 \quad \text{for} \quad T_1 > T_2$$

Poisson device 1 supplies net power to the circuit. By conservation of energy, device 2 must dissipate net power. The dissipation could be verified directly by factoring out the T_2 exponential in Eq. (5.36) to start the calculation.

5.5 Poisson to Gaussian

This section considers a circuit with both a linear resistor (with a Gaussian noise model) and a diode (with a Poisson noise model), as in Fig. 5-5. The first order of business is to to check that no heat flows when the devices are at the same temperature, *i.e.*, when $T_1 = T_2$. This test is not considered by Chapter 4, where we first considered Poisson and Gaussian models describing devices in the same circuit.
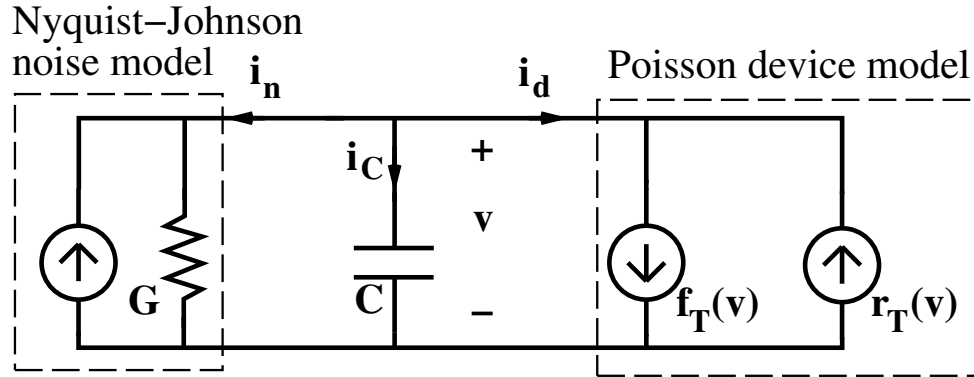
Figure 5-5: A diode and a linear resistor, at different temperatures, driving a capacitor

The expression

$\rho_n^o \, r_n \, E_n =$ steady-state rate at which the energy E_n is delivered by the reverse

source for transitions from n to $n + 1$

$=$ steady-state power delivered by the reverse source for transitions from

n to $n + 1$

was derived in Section 5.3. However, now the state space is continuous. Skipping over questions of uniqueness, the equilibrium density (for $T_1 = T_2$) must be

$$\rho_{eq}(q) = \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{q^2}{2kTC}\right]$$

by Thermodynamic Requirement #2. The rate $r_T(v_\beta(q))$ of jumps from q to $q + e$ was already defined for all voltages (because the capacitance need not have been an integer). The expression for the energy $E(q)$ of such a jump is also defined for continuous arguments. To find the total average power delivered by the reverse source, the term $\rho_n^o \, r_n \, E_n$ is integrated instead of summed.

$$P_r \;\; = \;\; \int_{-\infty}^{+\infty} \rho_{eq}(q) \, r_T(v_\beta(q)) \, E(q) \, dq$$

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{q^2}{2kTC}\right] \frac{I_S}{e} \frac{2qe + e^2}{2C} \, dq \\
&= \frac{I_S}{2C} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{q^2}{2kTC}\right] (2q + e) \, dq \\
&= \frac{I_S \, e}{2C},
\end{aligned}
$$

where we have again used the zero-mean property and normalization. Note that this is the same power computed in the discrete case: the addition of the Nyquist-Johnson noise source and resistor did not affect the power.

For the forward source,

$$
\begin{aligned}
P_f &= \int_{-\infty}^{+\infty} \rho_{eq}(q) \, f_T(v_\alpha(q)) \, (-E(q - e)) \, dq \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{q^2}{2kTC}\right] \frac{I_S}{e} \exp\left(\frac{(q - e/2)/C}{kT/e}\right) \left(-\frac{2qe - e^2}{2C}\right) \, dq \\
&= -\frac{I_S}{2C} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{1}{2kTC}(q^2 - 2qe + e^2)\right] (2q - e) \, dq \\
&= -\frac{I_S}{2C} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi kTC}} \exp\left[-\frac{1}{2kTC}\tilde{q}^2\right] (2\tilde{q} + e) \, d\tilde{q} \qquad\qquad (\tilde{q} = q - e) \\
&= -\frac{I_S \, e}{2C}.
\end{aligned}
$$

Again, the power into the forward and reverse Poisson sources cancel.

This cancellation result is not restricted to the diode model, because the net power can be calculated without knowing the constitutive relation for the device. (The power contributed by the forward and reverse sources cannot be calculated independently.) The net power delivered by the Poisson model is

$$
\begin{aligned}
P_{shot} &= P_r + P_f \\
&= \int_{-\infty}^{+\infty} \rho_{eq}(q) \, r_T(v_\beta(q)) \, E(q) \, dq + \int_{-\infty}^{+\infty} \rho_{eq}(\tilde{q}) \, f_T(v_\alpha(\tilde{q})) \, (-E(\tilde{q} - e)) \, d\tilde{q} \\
&= \int_{-\infty}^{+\infty} \rho_{eq}(q) \, r_T(v_\beta(q)) \, E(q) \, dq - \int_{-\infty}^{+\infty} \rho_{eq}(q + e) \, f_T(v_\alpha(q + e)) \, E(q) \, dq
\end{aligned}
$$

$$= \int_{-\infty}^{+\infty} \left[\rho_{eq}(q) \, r_T(v_\beta(q)) - \rho_{eq}(q+e) \, f_T(v_\alpha(q+e)) \right] E(q) \, dq. \qquad (5.38)$$

Recall that $v_\beta(q) = v_\alpha(q+e)$ and

$$\frac{f_T(v_\beta(q))}{r_T(v_\beta(q))} = \exp\left(v_\beta(q)/v_T\right) = \exp\left(\frac{q+e/2}{C} \frac{1}{kT/e}\right).$$

Also, because the equilibrium density must be the Gibbs distribution, the ratio of the values at two points separated by e is given by

$$\frac{\rho_{eq}(q+e)}{\rho_{eq}(q)} = \frac{\exp\left[\frac{-(q+e)^2}{2CkT}\right]}{\exp\left[\frac{-q^2}{2CkT}\right]} = \exp\left(-\frac{2qe+e^2}{2CkT}\right).$$

Therefore, the quantity in brackets in Eq. (5.38) is zero, and so is the integral:

$$P_{shot} = 0.$$

The same result must obtain for the power into the Nyquist-Johnson model: the power supplied by the current source is equal to that dissipated in the pure conductance. While the power dissipated in the conductor may be calculated using the equilibrium voltage density, the frequency-domain techniques of Section 5.2 for calculating the power supplied by the current source do not apply because of the nonlinear diode.

We would next like to check that there is power flow between the two models when the temperatures are not equal. Unfortunately, we do not know the steady-state distribution for this case, nor do the equations appear to show that the integrand is always positive.

Chapter 6

Limits to the Fluctuation-Dissipation Theorem for Nonlinear Circuits

This chapter diverges significantly from the subject of the rest of this thesis. The standard fluctuation-dissipation theorem for circuits relates the voltage or current fluctuations of a linear, time-invariant circuit to its impedance (or admittance). The previous chapters have all been concerned with nonlinear dissipative devices. In this chapter, we instead retain linear dissipative devices, but let the energy storage devices be nonlinear.

The material of this chapter has been accepted for publication as "Limits to the Fluctuation-Dissipation Theorem for Nonlinear Circuits" in *IEEE Trans. Circuits Syst. I* [51]. Minor changes have been made to integrate it into the thesis.

6.1   Introduction

Consider the bridge circuit of Figure 6-1. It is a standard result of linear circuit theory that under the matching condition $L = R^2 C$, the driving-point impedance
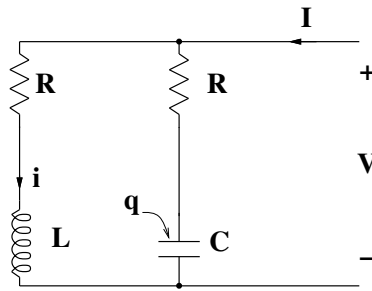
Figure 6-1: Linear noise-free bridge circuit is matched and has input impedance R if $L = R^2C$.

reduces to R and the natural frequency of the circuit does not appear as a pole [52, 53]. Regardless of the values of the capacitor and inductor, for high frequencies, the capacitor is essentially a short circuit, whereas the inductor is essentially an open circuit; at low frequencies, the opposite occurs. The matching condition ensures that a balance is preserved for intermediate frequencies: the charging of the capacitor is matched by the fluxing of the inductor. In the language of control theory, the state equations become *nonminimal* in the matched case.

Central Questions in this Chapter

Suppose one has two black boxes, one with a matched bridge circuit inside and the other with a single equivalent linear resistor. Is it possible to distinguish the two using the noise behavior? How does the answer change if the inductor and capacitor are nonlinear or time-varying?

The LTI Case

It is straightforward to verify directly in the LTI case that if a Nyquist-Johnson noise model [1, 2] (as shown in Fig. 6-2) is associated with each resistor, then the spectrum of the short-circuit terminal current in a matched bridge circuit is also that of a
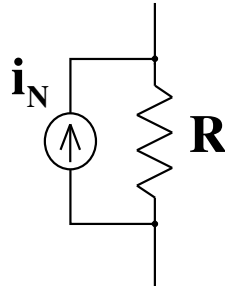
Figure 6-2: Nyquist-Johnson thermal noise model (Norton form) is a noiseless linear resistor in parallel with a Gaussian white noise current source i_N with power spectral density $2kT/R$.

Nyquist-Johnson noise model for a single resistor of value R. The verification can be done by standard frequency-domain techniques or by stochastic calculus [54]. The highpass filtering of the RC branch is precisely balanced by the lowpass filtering of the RL branch, so that the terminal noise spectrum is flat. Of course, both resistors must be at the same temperature. As noted in [53], applying a d.c. voltage to the circuit would result in differential heating of the resistor in the RL branch. If the resistors were not properly connected to thermal reservoirs, one could heat up and become noisier than the other, and the noise spectrum would no longer be flat. This is a trivial exception to the results of this chapter, which assumes uniform, constant temperature.

The result above is a particular example of a general circuit theory result, namely, that a one-port network of LTI passive elements with port admittance $Y(j\omega)$ presents a short-circuit thermal noise current with power spectrum $2kT \operatorname{Re}\{Y(j\omega)\}$, where k is Boltzmann's constant and T is the absolute temperature [55]. Physicists regard such results as particular cases of the fluctuation-dissipation theorem [25].

Generalizations in this Chapter

This chapter studies one carefully chosen example, motivated by the question of whether some form of fluctuation-dissipation theorem holds for some class of nonlinear circuits. Our initial formulation appears below as a conjecture for any pair of two-terminal networks, each comprising an interconnection of LTI resistors at a uniform, constant temperature, described by the Nyquist-Johnson model, and possibly also capacitors and inductors that may be nonlinear or time-varying. Two such networks are said to be *zero-state deterministically equivalent* if every applied terminal voltage waveform $v(t), t \geq 0$, produces the same current response $i(t)$ from both networks, provided all capacitor voltages and inductor currents are initially zero and all noise sources in the resistor models are set to zero. (In the LTI case this just means the two input admittances are identical.)

> *Preliminary Fluctuation-Dissipation Conjecture for Networks:*
> No two zero-state deterministically equivalent networks can be distinguished by their terminal noise current responses to any applied voltage waveform.

The conjecture just hypothesizes that the deterministic terminal behavior uniquely determines the noise current response for all voltage drives, independent of the details of the network. The conjecture is true in the LTI case. (Closely related formulations for the current-driven and multiport cases [55] also hold true for LTI networks, but we ignore them here for simplicity.)

Main Result of the Chapter

An examination of the bridge circuit will show that this preliminary conjecture is wrong when the applied voltage waveform or the circuit elements are time-varying.

This chapter considers only the Nyquist-Johnson model for noise in a linear resistor. That model does not assume any knowledge of the deterministic current flow mechanism. The results of this chapter disprove the existence of such "black-box" noise models for systems with internal nonlinearities when the nonlinearities are in the lossless subsections.

In Section 6.2 we develop the matching condition for the bridge circuit with nonlinear, time-invariant inductor and capacitor under which it becomes deterministically equivalent to a single linear resistor R at the terminals. In Section 6.3 we show that such a matched nonlinear bridge gives a short-circuit port current noise statistically identical to that of the Nyquist-Johnson model for R at thermal equilibrium. We also show that the same result holds for any d.c. applied voltage once the capacitor and inductor have settled to statistical steady-state. In Section 6.4 we develop the matching condition for the bridge circuit with linear time-varying inductor and capacitor. We show that in this case, however, the current noise is *not* that of the Nyquist-Johnson model for such a resistor, and thus the preliminary fluctuation-dissipation conjecture must be modified. We then apply this result to the nonlinear time-invariant bridge circuit linearized about any trajectory to conclude that the preliminary fluctuation-dissipation conjecture also fails for the nonlinear bridge circuit with time-varying input voltage.

All derivations are exact, involving no approximations, except for the last. Further details, including a stochastic calculus derivation for the LTI bridge, treatment of a dual circuit, and more explicit calculations in some proofs have been omitted here for brevity but can be found in [54]. Other mathematically-oriented studies of noise in nonlinear circuits include [9, 22, 28, 37].
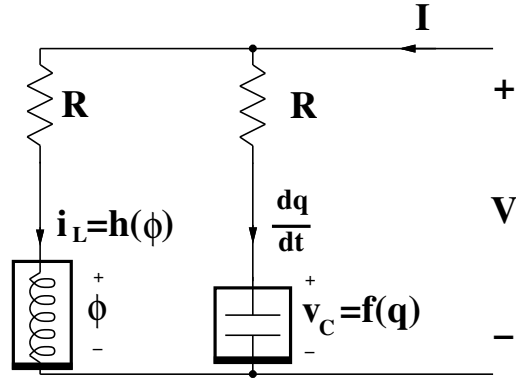
Figure 6-3: The nonlinear bridge circuit

6.2 Nonlinear, Noise-Free Case

Consider the circuit of Figure 6-3. Of course, $R > 0$. In addition, we require the following constraints, drawn essentially from [56, 57]:

Assumption 1: Nonlinear reactive element properties. The mappings $h : \phi \rightarrow i_L$ and $f : q \rightarrow v_C$ obey

(i) $h(0) = 0, f(0) = 0$

(ii) h and f are continuously differentiable functions, and for all values of the arguments and some fixed $\epsilon > 0$, there holds

$$\frac{dh}{d\phi} \geq \epsilon > 0 \qquad \text{and} \qquad \frac{df}{dq} \geq \epsilon > 0.$$

This assumption ensures that the circuit is passive, and that $(q, \phi) = (0, 0)$ is a globally asymptotically stable equilibrium point for $V = 0$.

As noted in Section 6.1, in the linear case the condition $L = R^2 C$ ensures that the bridge appears as a simple linear resistor at its terminals. In the following theorem, this condition is generalized by finding a condition relating the two nonlinearities which ensures this simple terminal behavior.

Theorem 1: Matching Condition for the Nonlinear Bridge. Consider the circuit of Figure 6-3, with Assumption 1 holding. Suppose the circuit is in the zero state at $t = 0$ and is excited by a voltage $V(t)$ for $t > 0$. Then for all $V(t)$ there holds

$$V(t) = R\,I(t) \tag{6.1}$$

for all $t \geq 0$, if and only if

$$f(q) = R\,h(Rq) \tag{6.2}$$

for all values of q.

Remark: Since $f'(q) = 1/C(q)$ is the reciprocal of the incremental capacitance and $h'(\phi) = 1/L(\phi)$ is the reciprocal of the incremental inductance, then Eq. (6.2) implies $L(\phi) = R^2\,C(q)|_{q=\phi/R}$, a local version of the linear matching condition $L = R^2 C$.

Remark: The above theorem is almost certainly not novel. However, we are unaware of a reference.

Proof: The circuit differential equations are

$$\frac{dq}{dt} = \frac{V - f(q)}{R} \tag{6.3}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi), \tag{6.4}$$

and the port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R}. \tag{6.5}$$

First suppose Eq. (6.2) holds. Observe from Eqs. (6.3) and (6.4) that, irrespective of $V(\cdot)$,

$$
\begin{aligned}
\frac{d(\phi - Rq)}{dt} &= -R\,h(\phi) + f(q) \\
&= -R\,h(\phi) + R\,h(Rq) \\
&= -R\,h'(\xi)\,(\phi - Rq),
\end{aligned}
$$

where ξ lies between ϕ and Rq, by application of the Mean Value Theorem. It follows that

$$
\begin{aligned}
\frac{d}{dt}\,[\phi - Rq]^2 &= -2R\,h'(\xi)\,(\phi - Rq)^2 \\
&\leq -2R\,\epsilon\,(\phi - Rq)^2,
\end{aligned}
\tag{6.6}
$$

using Assumption 1. Since $\phi(0) = q(0) = 0$, then for all $t \geq 0$, $\phi(t) = R\,q(t)$. Thus the matching condition (6.2) together with Eq. (6.5) yields $I(t) = V(t)/R$ as required.

Conversely, if we suppose that $I(t) = V(t)/R$ for all t, then from Eq. (6.5),

$$
R\,h(\phi(t)) = f(q(t))
\tag{6.7}
$$

must hold for all t. In addition the two parallel branches give two distinct expressions for $V(t)$, also evident from Eqs. (6.3) and (6.4):

$$
V = R\,h(\phi) + \frac{d\phi}{dt} = f(q) + R\frac{dq}{dt}.
$$

In light of Eq. (6.7), the last equality yields
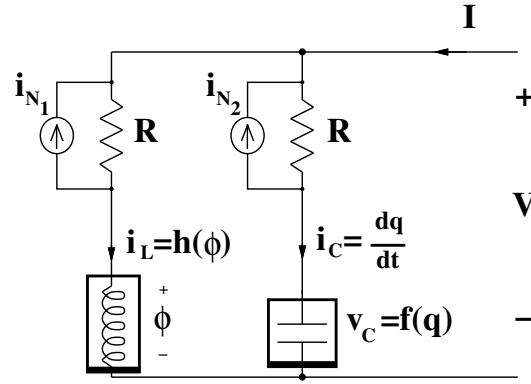
$$
\frac{d\phi}{dt} = R\,\frac{dq}{dt},
$$

Figure 6-4: Nonlinear bridge circuit with Nyquist-Johnson noise sources

and with zero initial conditions for ϕ and q, this means that

$$\phi(t) = R\,q(t). \tag{6.8}$$

Hence in Eq. (6.7), we have for all t, $R\,h(R\,q(t)) = f(q(t))$. Since all values of $q(t)$ are clearly attainable by using some appropriate $V(t)$, it follows that $R\,h(R\,q) = f(q)$ for all q, as required. ∎

Remark: The arguments above easily show that if the initial conditions are nonzero, then $\phi(t) - R\,q(t)$ decays to zero exponentially fast, and thus Eq. (6.1) holds asymptotically for large t.

6.3   Nonlinear, Noisy Case: Successful Results

For this section, a Norton-form Nyquist-Johnson noise model is associated with each resistor in the circuit, as in Figure 6-4. We would like to show that the terminal current noise of the matched bridge is the same as that for a single linear resistor, when V is constant and the circuit is in steady-state. To first order, this result is clear. Recall that the incremental capacitance and inductance satisfy $L(\phi) =$

$R^2 C(q)|_{q=\phi/R}$. A linearization about the noise-free equilibrium operating point (q, ϕ) for a d.c. applied voltage of a nonlinear matched circuit will yield a matched linear circuit. By superposition, the noise current for the linearized circuit is unaffected by the applied voltage. The point of this section is to show that this equivalence holds *exactly*, even for high temperatures or strong nonlinearities for which the noise could drive the circuit out of the valid region of linearization.

The circuit is described by stochastic differential equations (SDE's):

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \tag{6.9}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi) - R\,i_{N_1}, \tag{6.10}$$

where i_{N_1} and i_{N_2} are independent Gaussian white noise processes with power spectral density $2kT/R$. The port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R} - i_{N_2}. \tag{6.11}$$

One might be tempted to use the matching condition (6.7) and immediately conclude $I = V/R - i_{N_2}$. However, this condition does *not* hold, because (6.7) was derived for a different excitation: q and ϕ no longer satisfy $\phi(t) = R\,q(t)$, because they are now driven by independent noise sources. So, the power spectrum of I must be calculated more methodically.

6.3.1 The $I(t) - V(t)$ Relation in the Presence of Noise

Before proceeding to study the noise power spectrum, we show that the nonlinear inductor and capacitor cannot "rectify" the noise, even with a time-varying $V(t)$. Rectification would cause incorrect "average" behavior, or first-order statistics of the circuit, such that it would be pointless to study the second-order statistic of the power

spectral density.

> *Theorem 2: Terminal Noise Current is Zero-Mean.* Consider the circuit
> of Figure 6-4, described by equations (6.9) to (6.11), with Assumption
> 1 and the matching condition (6.2) in force. Let $V(t)$ be an arbitrary
> excitation, and assume zero initial conditions. Then

$$E\{I(t)\} = \frac{V(t)}{R}. \tag{6.12}$$

Proof: Taking expectations on both sides of Eq. (6.11),

$$E\{I\} = E\{h(\phi)\} + \frac{V(t)}{R} - \frac{E\{f(q)\}}{R} - 0. \tag{6.13}$$

In order to compute the expectations of $f(q)$ and $h(\phi)$, we need to know something about the probability densities ρ for q and ϕ. The Fokker-Planck equations [15, 20] for the evolutions of these densities are, for Eqs. (6.9) and (6.10), respectively,

$$\frac{\partial \rho_q}{\partial t} = -\frac{\partial}{\partial q}\left[\frac{V(t) - f(q)}{R} \rho_q\right] + \frac{kT}{R}\frac{\partial^2 \rho_q}{\partial q^2} \tag{6.14}$$

$$\frac{\partial \rho_\phi}{\partial t} = -\frac{\partial}{\partial \phi}\left[\Big(V(t) - R\,h(\phi)\Big)\rho_\phi\right] + kTR\frac{\partial^2 \rho_\phi}{\partial \phi^2}. \tag{6.15}$$

Using the matching condition (6.2), these two equations become identical up to a scaling. The reader can verify that a density $\rho_\phi(\phi, t)$ satisfies Eq. (6.15) if and only if the scaled version

$$\rho_q(q, t) = R\,\rho_\phi(R\,q, t) \tag{6.16}$$

satisfies Eq. (6.14). The densities corresponding to zero initial conditions (delta functions) also satisfy Eq. (6.16) at $t = 0$. Thus, the solutions of Eqs. (6.14) and (6.15) satisfy Eq. (6.16) for all time, and it follows by direct calculation that

$$E\{f(q(t))\} = R\,E\{h(\phi(t))\}, \quad t \geq 0. \tag{6.17}$$

Substituting Eq. (6.17) into Eq. (6.13) shows that the desired result (6.12) holds.   ∎

More details are given in [54].

Definition: A *steady-state* density satisfies $\frac{d\rho}{dt} = 0$. *Thermal equilibrium* for this circuit is the steady state with $V = 0$.

Corollary: Theorem 2 remains true if, instead of zero initial conditions, the circuit initially has a steady-state density with $V(0) \neq 0$.

Proof: The densities

$$\rho_q(q) = A_q \exp\left[\frac{1}{kT} \int_0^q (V - f(\tilde{q}))\, d\tilde{q}\right] \tag{6.18}$$

$$\rho_\phi(\phi) = A_\phi \exp\left[\frac{1}{kT} \int_0^\phi \left(\frac{V}{R} - h(\tilde{\phi})\right) d\tilde{\phi}\right], \tag{6.19}$$

where A_q and A_ϕ are normalization constants, are the steady-state solutions to Eqs. (6.14) and (6.15). Under the matching condition (6.2), the steady-state initial densities satisfy Eq. (6.16) at $t = 0$. Thus, the solutions of Eqs. (6.14) and (6.15) again satisfy Eq. (6.16) for all time, and the desired result (6.12) holds.   ∎

6.3.2   Thermal Noise Current

This section derives the thermal noise current spectrum at the external terminals of the circuit.

Theorem 3: Terminal Noise Current is that of a Nyquist-Johnson Resistor. Consider the circuit of Figure 6-4, described by Eqs. (6.9) to (6.11) with Assumption 1 and the matching condition (6.2) in force. Assume the circuit is in steady-state at $t = 0$. Denote by R_{nn} the autocorrelation of the terminal noise current $n(t) = I(t) - V(t)/R$. Then for $t, \tau > 0$,

(a) $E\{n(t)\} = 0$,

(b) $R_{nn}(t - \tau) = \frac{2kT}{R}\,\delta(t - \tau)$, and

(c) $\int_0^t n(s)\,ds$ is a scaled Wiener process,

provided that one of the following two sufficient conditions holds:

(i) the circuit is LTI, *i.e.*, $f(q) = q/C$ and $h(\phi) = \phi/L$, or

(ii) the voltage $V(t)$ is constant.

Proof:

(i) The sufficiency of condition (i) is an immediate consequence of superposition for linear circuits. The deterministic behavior was shown in Section 6.2, and the noise behavior for linear circuits at equilibrium was shown in [55]. Adding together the results of the independent excitations proves the theorem for this condition.

(ii) $\Rightarrow$ (a) This was shown in Theorem 2.

(ii) $\Rightarrow$ (b) The autocorrelation of $n(t)$ is, from Eq. (6.11),

$$R_{nn}(t, \tau) = E\left\{ \left[h(\phi(t)) - \frac{f(q(t))}{R} - i_{N_2}(t) \right] \left[h(\phi(\tau)) - \frac{f(q(\tau))}{R} - i_{N_2}(\tau) \right] \right\}.$$

Since $i_L(\cdot)$ is independent of $i_{N_2}(\cdot)$ and the latter has zero mean,

$$E\left\{ h(\phi(t))\, i_{N_2}(\tau) \right\} = E\left\{ h(\phi(t)) \right\} E\left\{ i_{N_2}(\tau) \right\} = 0.$$

Since $i_L(\cdot)$ is also independent of $v_C(\cdot)$, though neither has zero mean,

$$E\left\{ h(\phi(t)) \frac{f(q(\tau))}{R} \right\} = E\left\{ h(\phi(t)) \right\} E\left\{ \frac{f(q(\tau))}{R} \right\}.$$

The proof of Theorem 2 used the similarity of the Fokker-Planck equations (6.14) and (6.15) to show

$$E\left\{ f(q(t)) \right\} = R\, E\left\{ h(\phi(t)) \right\},$$

for all times t (or τ), and similarly there holds

$$E\Big\{h(\phi(t))\,h(\phi(\tau))\Big\} = E\Big\{\frac{f(q(t))}{R}\,\frac{f(q(\tau))}{R}\Big\}.$$

The autocorrelation can thus be simplified to

$$\begin{aligned}
R_{nn}(t,\tau) &= 2E\Big\{\frac{f(q(t))}{R}\,\frac{f(q(\tau))}{R}\Big\} - 2E\Big\{\frac{f(q(t))}{R}\Big\}\,E\Big\{\frac{f(q(\tau))}{R}\Big\}\\
&\quad + E\Big\{\frac{f(q(t))}{R}\,i_{N_2}(\tau)\Big\} + E\Big\{i_{N_2}(t)\,\frac{f(q(\tau))}{R}\Big\} + E\Big\{i_{N_2}(t)\,i_{N_2}(\tau)\Big\}. \quad (6.20)
\end{aligned}$$

If we multiply both sides of the differential equation (6.9) for $q(t)$ by $f(q(\tau))$ and take expectations, we obtain

$$\frac{d}{dt}E\Big\{f(q(\tau))\,q(t)\Big\} = \frac{V(t)}{R}\,E\Big\{f(q(\tau))\Big\} - \frac{1}{R}E\Big\{f(q(t))\,f(q(\tau))\Big\} - E\Big\{i_{N_2}(t)\,f(q(\tau))\Big\}.$$
$$(6.21)$$

The dummy time indices t and τ may be interchanged, corresponding to writing the SDE in τ and multiplying through by $f(q(t))$, to get

$$\frac{d}{d\tau}E\Big\{f(q(t))\,q(\tau)\Big\} = \frac{V(\tau)}{R}\,E\Big\{f(q(t))\Big\} - \frac{1}{R}E\Big\{f(q(\tau))\,f(q(t))\Big\} - E\Big\{i_{N_2}(\tau)\,f(q(t))\Big\}.$$
$$(6.22)$$

Define

$$F(t,\tau) = E\Big\{f(q(\tau))\,q(t)\Big\},$$

so that the autocorrelation may be expressed

$$\begin{aligned}
R_{nn}(t,\tau) &= \Big[\frac{V(t)}{R} - E\Big\{\frac{f(q(t))}{R}\Big\}\Big]\,E\Big\{\frac{f(q(\tau))}{R}\Big\}\\
&\quad + \Big[\frac{V(\tau)}{R} - E\Big\{\frac{f(q(\tau))}{R}\Big\}\Big]\,E\Big\{\frac{f(q(t))}{R}\Big\}\\
&\quad - \frac{1}{R}\Big[\frac{dF(t,\tau)}{dt} + \frac{dF(\tau,t)}{d\tau}\Big] + E\Big\{i_{N_2}(t)\,i_{N_2}(\tau)\Big\}. \quad (6.23)
\end{aligned}$$

For arbitrary time-varying $V(t)$ and strictly nonlinear inductor and capacitor, no further simplification is apparent.

We now require condition (ii). Since V is constant and the system is initially at steady-state, it remains in steady-state for $t \geq 0$, *i.e.*, $q(t)$ and $\phi(t)$ are *stationary* random processes. Taking expectations of both sides of the differential equation (6.9),

$$E\left\{\frac{dq}{dt}\right\} = 0 = E\left\{\frac{V - f(q(t))}{R}\right\} + E\left\{i_{N_2}(t)\right\},$$

so that

$$V = E\left\{f(q(t))\right\}.$$

Since $q(t)$ is stationary, $F(t, \tau) = F(t - \tau)$ depends only on the difference $(t - \tau)$. Further, a consequence of Assumption 1 and Eq. (6.9) is that $q(t)$ is a *reversible* process [28], *i.e.*, for all t_1 and t_2,

$$\Pr\left[\alpha \leq q(t_1) \leq \alpha + d\alpha, \beta \leq q(t_2) \leq \beta + d\beta\right] =$$
$$\Pr\left[\beta \leq q(t_1) \leq \beta + d\beta, \alpha \leq q(t_2) \leq \alpha + d\alpha\right].$$

As a consequence of reversibility, F is an even function:

$$\begin{aligned} F(t - \tau) &= E\left\{q(t)\, f(q(\tau))\right\} = \iint af(b)\, p(q(t) = a, q(\tau) = b)\, da\, db \\ &= \iint af(b)\, p(q(\tau) = a, q(t) = b)\, da\, db = E\left\{q(\tau)\, f(q(t))\right\} \\ &= F(\tau - t), \end{aligned} \tag{6.24}$$

where $p(\cdot\,,\,\cdot)$ represents the joint probability density of its two arguments, and equality between the first and second lines follows from reversibility. Since $F(\cdot)$ is an even function, $F'(\cdot)$ must be odd, and

$$\frac{d}{dt}F(t, \tau) = \frac{d}{dt}F(t - \tau) = F'(t - \tau) = -F'(\tau - t) = -\frac{d}{d\tau}F(\tau, t). \tag{6.25}$$

Therefore, the autocorrelation reduces to

$$R_{nn}(t, \tau) = E\left\{i_{N_2}(t)\, i_{N_2}(\tau)\right\} = \frac{2kT}{R}\, \delta(t - \tau).$$

(ii) $\Rightarrow$ (c) It remains to show that $w(t) \overset{\Delta}{=} \int_0^t n(s)\, ds$ is a scaled Wiener process, or equivalently, that $n(t)$ is a Gaussian white noise process. From the zero-mean property of $n(t)$ and its covariance, it is trivial to see that $w(t)$ obeys $E\left\{w(t)\right\} = 0$ and $E\left\{w(t)\, w(s)\right\} = \frac{2kT}{R}\min[t, s]$, and $w(t)$ is a martingale.[1] From Eqs. (6.9) and (6.10), it follows that the sample paths of ϕ and q are continuous with probability 1, by a result of stochastic differential equation theory [15], and accordingly from an integrated version of Eq. (6.11), $w(t)$ also has this property. A theorem of Doob [45] then allows one to conclude that because $w(t)$ is a continuous martingale with covariance equal to that of a scaled Wiener process, it is necessarily itself a scaled Wiener process. $\blacksquare$

It is perhaps somewhat surprising that this analysis holds exactly. There are two noise sources driving nonlinear elements, so one might expect a nonlinear "mixing" under which the two drives interact to produce a colored noise spectrum, but this does not happen in this circuit.

6.4   Failures of the Conjecture

As mentioned in the introduction, there are some situations in which the noise current of the matched bridge circuit is not statistically equivalent to the noise of a single linear resistor. Even if the circuit is kept at constant temperature, the conjecture fails for a time-varying circuit. This failure casts doubts on the hopes of establishing the general nonlinear nonequilibrium result for a time-varying driving voltage.

---

[1]A martingale is a random process $w(t)$ such that the conditional expectation for the future, given the entire past, is simply the present value. Symbolically, $E\left\{w(t_2) \mid w(t), 0 \le t \le t_1\right\} = w(t_1)$ for all $t_2 > t_1$.

Suppose the energy storage elements in Figure 6-4 are linear, but time varying. This will provide the first nontrivial failure of fluctuation-dissipation hypothesis in the Introduction; it is sufficient to consider the short-circuit (undriven) behavior. The circuit differential equations are

$$\frac{d\phi}{dt} = -\frac{R\,\phi(t)}{L(t)} - R\,i_{N_1}(t) \tag{6.26}$$

$$\frac{dq}{dt} = -\frac{q(t)}{R\,C(t)} - i_{N_2}(t), \tag{6.27}$$

and we assume $E\{q(0)\} = E\{\phi(0)\} = 0$ so that $q(t)$ and $\phi(t)$ are zero mean. The port current is

$$I(t) = \frac{\phi(t)}{L(t)} - \frac{q(t)}{R\,C(t)} - i_{N_2}(t). \tag{6.28}$$

The corresponding matching condition is of course

$$L(t) = R^2\,C(t). \tag{6.29}$$

The differential equation for $q(t)$ can be solved explicitly in terms of sample paths of the noise process $i_{N_2}(t)$:

$$q(t) = \exp\left[-\int_0^t \frac{ds}{RC(s)}\right]\left(q(0) - \int_0^t i_{N_2}(\sigma)\,\exp\left[\int_0^\sigma \frac{ds}{RC(s)}\right]d\sigma\right). \tag{6.30}$$

The autocorrelation function for the port current (which is entirely noise current) for $\tau > t$ is

$$
\begin{aligned}
R_{nn}(t,\tau) &= E\left\{\left[\frac{\phi(t)}{L(t)} - \frac{q(t)}{R\,C(t)} - i_{N_2}(t)\right]\left[\frac{\phi(\tau)}{L(\tau)} - \frac{q(\tau)}{R\,C(\tau)} - i_{N_2}(\tau)\right]\right\} \\
&= \frac{1}{R^2 C(t)C(\tau)}E\left\{q(t)\,q(\tau)\right\} + \frac{1}{L(t)L(\tau)}E\left\{\phi(t)\,\phi(\tau)\right\} \\
&\quad + \frac{1}{RC(\tau)}E\left\{q(\tau)\,i_{N_2}(t)\right\} + E\left\{i_{N_2}(t)\,i_{N_2}(\tau)\right\}, \tag{6.31}
\end{aligned}
$$

where the other terms vanish because the variables are uncorrelated as argued previously but now also zero-mean, or by causality in that $i_{N_2}(\tau)$ cannot affect $q(t)$ for $\tau > t$. Again by appeal to the Fokker-Planck equations and the matching condition (6.29), it can be shown that

$$\frac{1}{R^2\, C(t)\, C(\tau)}\, E\Big\{q(t)\, q(\tau)\Big\} = \frac{1}{L(t)\, L(\tau)}\, E\Big\{\phi(t)\, \phi(\tau)\Big\}.$$

Thus, in order that the short-circuit current noise have the proper autocorrelation, it must be shown that

$$\frac{2}{R^2\, C(t)\, C(\tau)} E\Big\{q(t)\, q(\tau)\Big\} + \frac{1}{RC(\tau)}\, E\Big\{q(\tau)\, i_{N_2}(t)\Big\} = 0. \qquad (6.32)$$

Two quick calculations from Eq. (6.30) yield

$$
\begin{aligned}
E\left\{q(t)\, q(\tau)\right\} = {} & \exp\left[-\int_0^t \frac{ds}{RC(s)}\right] \exp\left[-\int_0^\tau \frac{1}{RC(s)}ds\right] \\
& \times \left(E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma\right)
\end{aligned}
$$

and

$$E\left\{q(\tau)\, i_{N_2}(t)\right\} = -\frac{2kT}{R}\exp\left[-\int_0^\tau \frac{ds}{RC(s)}\right] \exp\left[\int_0^t \frac{ds}{RC(s)}\right].$$

Substituting these into Eq. (6.32) and canceling common factors, the test reduces to

$$0 \overset{?}{=} E\left\{q^2(0)\right\} + \frac{2kT}{R}\int_0^t \exp\left[2\int_0^\sigma \frac{ds}{RC(s)}\right] d\sigma - C(t)\, kT \exp\left[2\int_0^t \frac{ds}{RC(s)}\right]. \quad (6.33)$$

Differentiating by t will yield a necessary condition for the equation to be true:

$$
\begin{aligned}
0 \overset{?}{=} {} & \frac{2kT}{R}\exp\left[2\int_0^t \frac{ds}{RC(s)}\right] d\sigma - \frac{dC(t)}{dt}kT \exp\left[2\int_0^t \frac{ds}{RC(s)}\right] \\
& - C(t)\, kT \exp\left[2\int_0^t \frac{ds}{RC(s)}\right]\left(\frac{2}{RC(t)}\right)
\end{aligned}
$$

$$= -\frac{dC(t)}{dt} \, kT \, \exp\left[2\int_0^t \frac{ds}{RC(s)}\right].$$

Thus, the time-varying bridge does not have stationary current noise at the terminals as required by the Nyquist-Johnson model, except in the trivial case that C is a constant. In this case, the integrals in Eq. (6.33) can be computed, and if the system starts at equilibrium, $i.e.$, $E\{q^2(0)\} = kTC$, then this condition is sufficient as well as necessary. Of course, if C is a constant, then the bridge is simply the standard linear, time-invariant circuit, for which the result was already known.

Remark: For a driving voltage $V(t)$ significantly larger than the noise, one could solve the deterministic system and then compute an approximation for the noise behavior by linearization about this time-varying solution. This approximation would behave like the time-varying linear system described above. Since the second-order statistics for that system are incorrect, we believe that the second-order statistics for the nonlinear system driven by a time-varying voltage will not match the statistics of a single linear resistor driven by that same voltage.

Chapter 7

Conclusions

7.1 Results in this Thesis

This thesis has presented four specific requirements that determine whether a noise model is acceptable. All are based on the second law of thermodynamics. They provide guidelines for developing physically correct device noise models to correspond with experimental data.

One important underlying assumption is that the behavior of the device during *any* equilibrium fluctuations is constrained by thermodynamic principles. At thermal equilibrium, the voltage and current fluctuations are generally small and the nonlinear device behavior could be approximated by linearizing about the origin of the $v - i$ curve. But on rare occasions, the fluctuations will be large enough to briefly drive the device into the nonlinear regime. The Gibbs distribution assigns to these fluctuations very small probabilities, which may not be experimentally measurable. However, models that predict non-thermodynamic behavior during large fluctuations (however rare) are *non-physical* and should be abandoned.

The Nyquist-Johnson Gaussian thermal noise model for linear resistors, extended to include nonequilibrium operating conditions, satisfies all three of these thermody-

namic requirements. In contrast, even the equilibrium requirements *cannot* be met by the Gaussian model for *any* nonlinear element with *any* choice of (operating-point dependent) noise amplitude.[1] In particular, the Gaussian noise model obtained by applying the Nyquist-Johnson formula to the linearized conductance, *e.g.* (2.47), is physically incorrect except in the short-circuit case, though it occasionally appears in the literature.

We have derived a constraint (2.38) under which the shot-noise model satisfies all thermodynamic requirements presented here, when describing a nonlinear device connected to a capacitor. This constraint allows one to predict the current-noise amplitude at every operating point *from knowledge of the device's $v - i$ curve alone.* The familiar subthreshold MOSFET and *pn* junction models satisfy this constraint.

Further, we required that a noise model for a device not depend on the circuit to which the device is connected. For our two-terminal, voltage-controlled resistive elements, the simplest tests involved only a capacitor connected to our device. In this case, the current noise may not depend on the value of the capacitance nor on the total charge accumulated on the capacitor plates, but only on the voltage across the capacitor's terminals. This is indeed exactly what one normally means by "device model." This requirement was fundamental in the interpretation of Eqs. (2.19), (2.20), and (2.32). However, our Poisson model does not satisfy this requirement for finite electron sizes.

In the limit that the electron charge goes to zero, the Poisson model no longer depends on the capacitor. The dependence for finite electron charge is a quantum-mechanical effect, since the electron is a charge quantum. In quantum-mechanical systems, one cannot expect to simply combine equations for the subsystems to describe the interconnection.

The comparison in Section 2.6 showed that one cannot determine whether a noise

---

[1]at least in the standard Itô and Stratonovich interpretations

model is thermodynamically acceptable by examining its power spectral density alone. Surprisingly, models with two different noise amplitudes (and different underlying statistics) turn out to be thermodynamically acceptable for a linear resistor. The power spectral density is neither sufficient nor necessary; further information on the underlying probability distribution is required. However, the experiments of Appendix A show that the linear resistor is not a device that can be described by the Poisson model. Appendix D shows the difficulties that would face an experimentalist attempting to find conditions under which the Poisson model for shot noise will differ appreciably from the predictions of the Gaussian model applied to a linearized conductance.

We have explored an extension of the fluctuation-dissipation theorem (or, in circuit theory terms, a result relating impedances to noise spectra) to a nonlinear situation. The spectral calculations have been nontrivial, calling on a reversibility idea and martingale theory. The positive results hold for a specific time-invariant bridge circuit, linear or nonlinear, in thermal equilibrium or at d.c. steady-state.

The negative results in Section 6.4 show that our original fluctuation-dissipation conjecture is not correct as stated and must be limited to exclude time-varying networks and nonlinear networks with time-varying inputs. Is the modified form below correct? This remains an open question in the field, and some of the ideas in [28] may be of assistance.

> *Modified Fluctuation-Dissipation Conjecture for Circuits*
> No two zero-state deterministically equivalent *time-invariant* networks can be distinguished by the terminal noise currents *at any d.c. voltage input* when the networks are in statistical steady-state.

The assumptions here remain those in the paragraph preceding the initial formulation (see the Introduction), including LTI Nyquist-Johnson resistors and nonlinear inductors and capacitors. Additional assumptions may be required to guarantee re-

versibility of the charge or flux random processes. The further extensions to include nonlinear resistor noise models or multiterminal circuits remain completely unexplored, so far as we know.

7.2 Suggestions for Further Work

Although the Poisson shot-noise model has been well established by this work, there are still many questions that could be answered.

One critical question is: how does one determine what devices can be described by Poisson models? What is it about the devices that allows this description? It is not the existence of the potential barrier [58]; this is supported by the idea that the noise in a diode is really generation-recombination noise [48]. To test this idea, one would look for devices with noisy generation-recombination processes but no potential barriers.

Since bipolar transistors consist of two pn junctions, it might be possible to extend the work of this paper to multiterminal Poisson models.

From a duality perspective, it is unsatisfying that there is not a dual shot-noise model that injects quanta of voltage or flux. We believe that the mathematics would still work out: in fact, putting the present shot-noise model for charge quanta on the other side of a gyrator would give the same result. A Josephson junction has quanta of flux, but it is superconducting, hence not a dissipative device. We are unaware of a physical device on which to base our model

One fundamental physical question is: is there a transport-level derivation of Nyquist Johson noise? This derivation would describe the motion of electrons, either by random diffusion or drift in an applied field, in the same formulation. If one divides the resistor into along its length, and varies the transmission probabilities across the boundaries of the sections, one might get shot noise out of the same derivation. By decreasing the doping level of pn junctions, one eventually gets bulk silicon, which

is a (linear) resistive material. This logic would not hold if shot noise in a diode is really generation-recombination noise. Thermal noise is about agitation of electrons in the conduction band, which is physically different from generation-recombination processes that add or remove electrons from the conduction band.

From a mathematics standpoint, the fact that both the Gaussian and the Poisson processes have maximal entropy under some conditions is quite intriguing. It is well known that the sum of two (independent) Gaussians is again a Gaussian; perhaps less well-known that the sum of two independent Poisson processes is again a Poisson process [13]. Loève's book [59] has some interesting theory of the central limit theorem as it applies to Poisson random variables.

Last but not least, the proper conditions on an initial distribution must be found that guarantee existence and uniqueness of solutions to the stochastic differential equations driven by point processes whose rates depend on the state of the system.

Appendix A

Simple Experiments

The measurements here show that the linear resistor does not obey the Poisson shot-noise model. This result is not particularly surprising, given that the Nyquist-Johnson model has stood sixty years. Thornber states in [60] that "owing to strict charge neutrality in the resistor, shot noise is not present," but it is not clear to us what this means. The Poisson model predicts a higher current noise for $V \neq 0$, whereas the extended Nyquist-Johnson model states that the noise is fixed for all voltages.

Recall in Section 2.6, we applied both models to a linear conductor G. The Poisson model (2.46) reduces to

$$S_{ii}^P = \frac{2e\,G\,V}{\tanh(V/2v_T)}, \tag{A.1}$$

while the Nyquist-Johnson model, of course, gives

$$S_{ii}^{NJ} = 4kTG. \tag{A.2}$$

(We have doubled the expressions in Section 2.6 for consistency with the measurements of the spectrum analyzer, which displays results for positive frequencies only.) For an applied voltage of only a few times v_T, the Poisson model predicts a doubling of the noise, compared with $V = 0$.
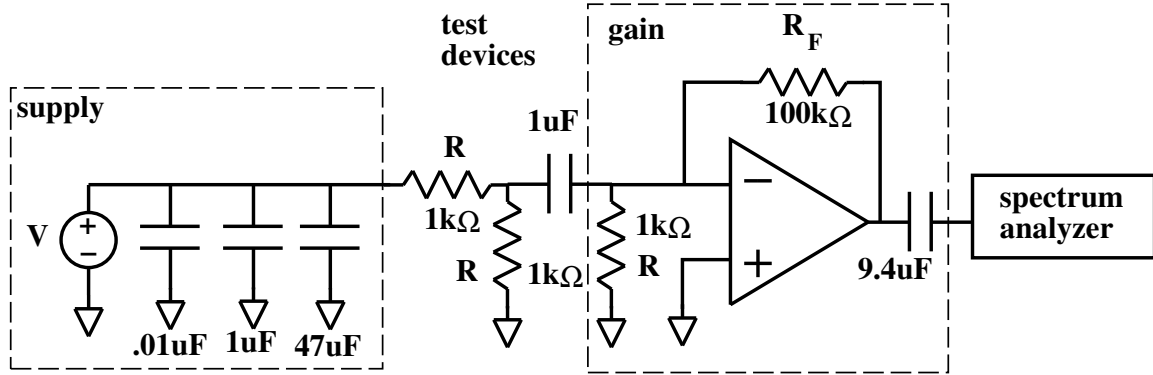
Figure A-1: Experimental setup.

Fig. A-1 diagrams the measurement circuit. The spectrum analyzer was a Hewlett-Packard 8568A. The amplifier was an AD829 operational amplifier. The gain of the circuit, determined by the two resistors in the box marked gain, is R_F/R, where $R_F = 100k\Omega$ and $R = 1k\Omega$. The capacitor between the op amp and the spectrum analyzer blocked DC signals (those below the corner frequency of about 340 Hz).

The capacitors on the left side of the circuit, inside the box marked supply, helped stabilize the voltage. $V = 6V$ was supplied by a 6V lantern battery; $V = 0V$ was achieved by connecting directly to ground.

For high-frequency noise signals, the test devices experience the same gain as the $1k\Omega$ resistor in the gain box. The capacitor blocks the DC component from reaching the op amp input.

According to the extended Nyquist-Johnson model for linear resistors, this circuit will have output voltage noise

$$\Delta V_{NJ} = \sqrt{\Delta V_{RF}^2 + \left(3\Delta I_R^2\right) R_F^2} = \sqrt{4kTR_F + 3\,\frac{4kT}{R}R_F^2}\ .$$

For the values in the circuit, $\Delta V_{NJ} = 7.1 \times 10^{-7} V/\sqrt{Hz}$.

For $V = 0V$, as seen in Section 2.6, the Poisson model predicts the same noise

as the Nyquist-Johnson model. For $V = 6$V, the voltage across each of the two test resistors is 3V, but the resistor in the gain box has no DC voltage across it, so

$$\Delta V_P = \sqrt{4kTR_F + \left(2\, \frac{2e \cdot 1/R \cdot 3}{\tanh(3/v_T)} + \frac{4kT}{R} \right) R_F^2} \; .$$

For this circuit, $\Delta V_P = 4.4 \times 10^{-6} V/\sqrt{Hz}$ at 6V, a factor of about 6 higher than the Nyquist-Johnson model.

The plots on the next page show that the output voltage noise does not change significantly at all. It is approximately 1×10^{-6} for both $V = 0$V and $V = 6$V. This is slightly higher than the predicted noise, due to contributions from the op amp and RF interference that were neglected. However, it is clearly less than the noise that should have been present if the Poisson model were correct.

In another experiment, the resistor values were increased to $R = 4k\Omega$; the measured voltage noise dropped to 4×10^{-7}. Theoretically, we expected this approximate halving of the noise, since the dominant term in ΔV_{NJ} is proportional to $\sqrt{1/R}$. This gives us some confidence that the experiment is measuring the noise we designed it to (rather than the noise of the op amp or the noise floor of the spectrum analyzer).

carbon film   V1=0, R =1k   15-Jul-1999 14:51

carbon film   V1=6, R =1k   15-Jul-1999 14:36
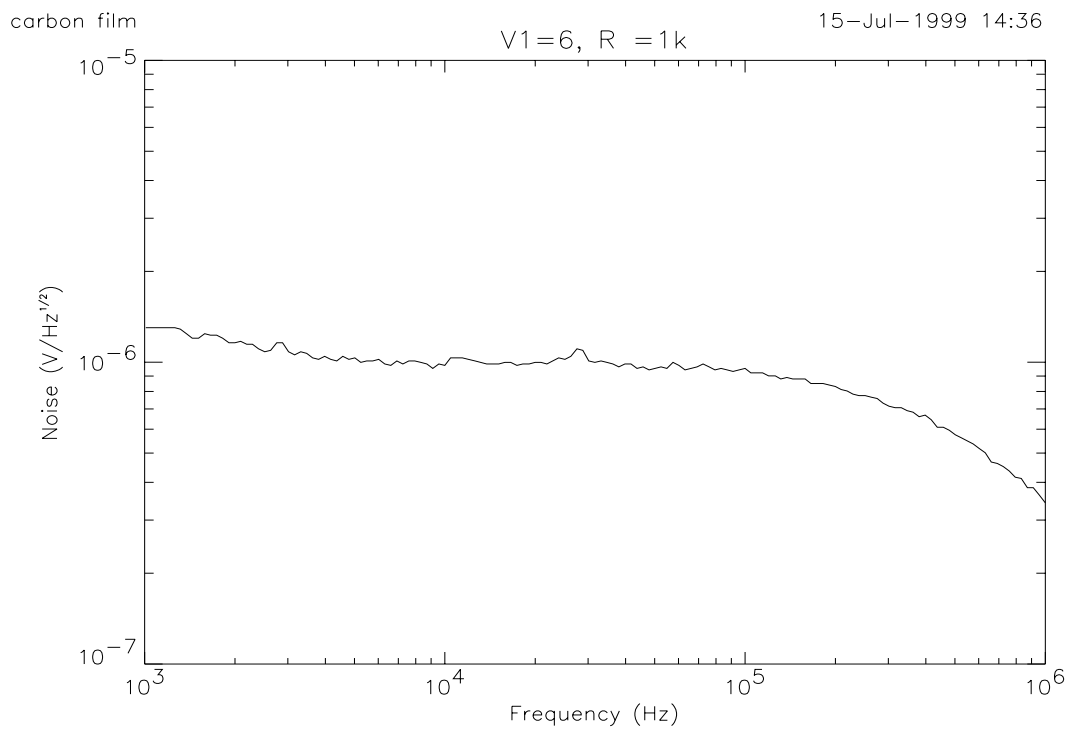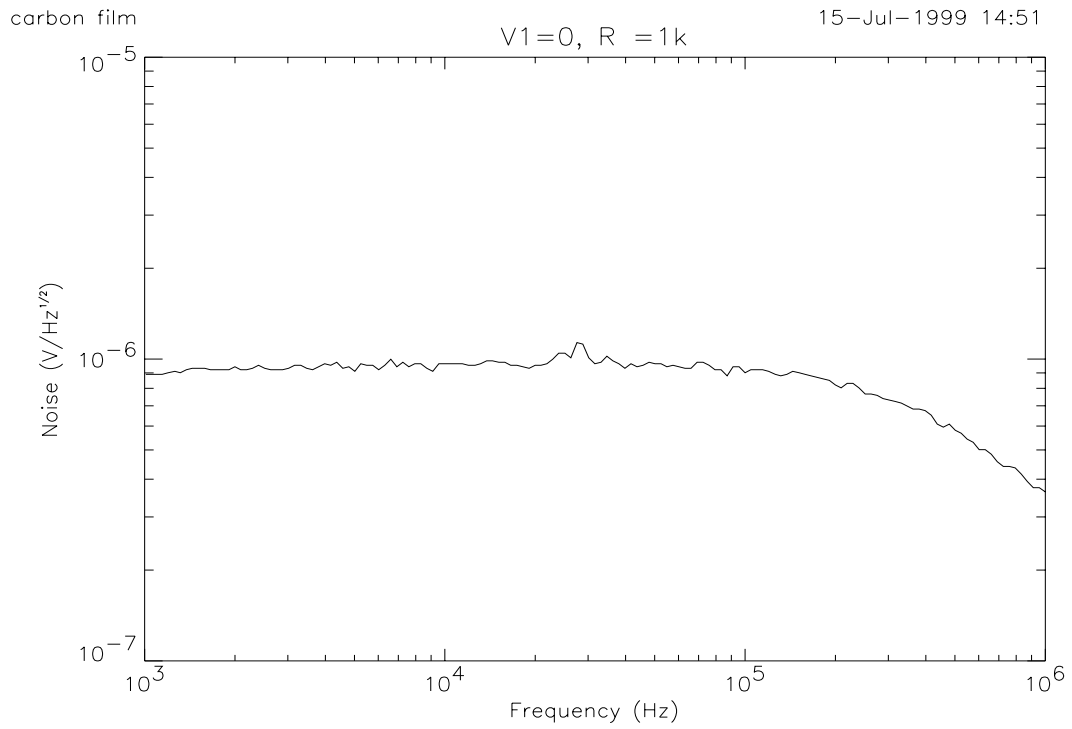
Appendix B

Time-Varying Linear Resistors

This Appendix considers the thermodynamic behavior of an LC circuit driven by a time-varying resistor. The analysis is motivated by the model for $1/f$ noise as a fluctuating resistance, or fluctuating transconductance in the channel of a MOS device [61]. The resistor will have a time-dependent resistance $R(t)$, and the noise current variance at each time τ will be that predicted by the Nyquist-Johnson model for a resistance $R(\tau)$.

Circuit Differential Equations

Consider the parallel-LC circuit configuration of Fig. B-1. The circuit differential equations, which follow from Kirchoff's Laws, are

$$
\frac{dq}{dt} \;=\; i_C = -h(\phi) - \frac{f(q)}{R(t)} - \sqrt{\frac{2\,kT}{R(t)}}\,\xi(t) \tag{B.1}
$$

$$
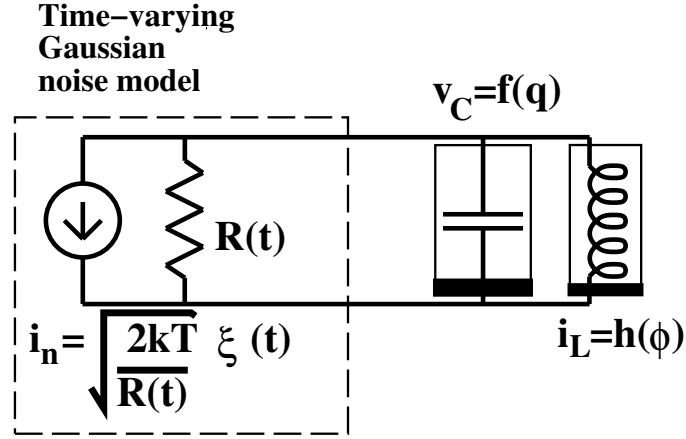\frac{d\phi}{dt} \;=\; v_L = v_C = f(q). \tag{B.2}
$$

Figure B-1: Time-varying resistor in an RLC circuit

The Fokker-Planck equation for the time-evolution of the probability density is therefore

$$
\begin{aligned}
\frac{d}{dt}\rho(t,q,\phi) \;=\; & \frac{\partial}{\partial q}\left[\left(h(\phi) + \frac{f(q)}{R(t)}\right)\rho(t,q,\phi)\right] - f(q)\frac{\partial}{\partial\phi}\Big[\rho(t,q,\phi)\Big] \\
& + \frac{\partial^2}{\partial q^2}\left[\frac{kT}{R(t)}\,\rho(t,q,\phi)\right].
\end{aligned}
\tag{B.3}
$$

The first two terms correspond to the drift, and the last term expresses diffusion.

Equilibrium Density

Following the steps in Chapter 2 for the constant resistor, the first test is to verify that the equilibrium density is the Gibbs distribution:

$$
\rho_{eq}(\mathbf{q},\boldsymbol{\phi}) = A\exp\left[-E_{LC}(\mathbf{q},\boldsymbol{\phi})/kT\right],
\tag{B.4}
$$

where E_{LC} is the energy stored in the capacitors and inductors, and A serves to normalize the equation. For an LC circuit, this is

$$\rho_{eq}(q, \phi) = A \exp \left[\frac{-\left(\int_0^\phi h(\tilde{\phi}) \, d\tilde{\phi} + \int_0^q f(\tilde{q}) \, d\tilde{q} \right)}{kT} \right].$$ (B.5)

The Fokker-Planck equation requires the partial derivatives of this density.

$$\frac{\partial}{\partial q} \rho_{eq}(q, \phi) = -\frac{f(q)}{kT} \rho_{eq}(q, \phi)$$ (B.6)

$$\frac{\partial}{\partial \phi} \rho_{eq}(q, \phi) = -\frac{h(\phi)}{kT} \rho_{eq}(q, \phi)$$ (B.7)

Plugging these derivatives in to Eq. (B.3),

$$
\begin{aligned}
0 \overset{?}{=} \; & h(\phi) \left(\frac{-f(q)}{kT} \right) \rho_{eq}(q, \phi) + \frac{\partial}{\partial q} \left[\frac{f(q)}{R(t)} \, \rho_{eq}(q, \phi) \right] - f(q) \left(\frac{-h(\phi)}{kT} \right) \rho_{eq}(q, \phi) \\
& + \frac{kT}{R(t)} \frac{\partial^2}{\partial q^2} \rho_{eq}(q, \phi) \\
= \; & \frac{1}{R(t)} \frac{\partial}{\partial q} \left[f(q) + kT \frac{-f(q)}{kT} \, \rho_{eq}(q, \phi) \right] = 0,
\end{aligned}
$$ (B.8)

we verify that the Gibbs distribution is an equilibrium for the circuit.

Increasing Entropy

In this section, the entropy will be shown to increase monotonically, without use of a closed-form solution for the probability density ρ.

The entropy of the energy storage side is classically defined as

$$S_{LC} = -k \iint \rho(t, q, \phi) \, \log \rho(t, q, \phi) \, dq \, d\phi$$ (B.9)

The time rate of change of the inductor-capacitor entropy is then

$$
\begin{aligned}
\frac{dS_{LC}}{dt} &= -k \iint \frac{d\rho}{dt} \log \rho \, dq \, d\phi - k \iint \rho \frac{1}{\rho} \frac{d\rho}{dt} \, dq \, d\phi \\
&= -k \iint \dot{\rho} \log \rho \, dq \, d\phi.
\end{aligned}
\tag{B.10}
$$

The second term from the product rule vanishes, because total probability is conserved. The reservoir entropy is calculated by use of the First Law,

$$
\frac{d\overline{E_{LC}}}{dt} = -T \frac{dS_R}{dt},
\tag{B.11}
$$

where $\overline{E_{LC}}$ is the expected energy stored in the inductor and capacitor, T is the reservoir temperature, and S_R is the entropy of the reservoir. Rearranging,

$$
\dot{S}_R = -\frac{1}{T} \iint E_{LC} \, \dot{\rho} \, dq \, d\phi
\tag{B.12}
$$

Adding up Eqs. (B.10) and (B.12), the rate of change of the total entropy is

$$
\dot{S}_{tot} = \dot{S}_{LC} + \dot{S}_R = \iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \dot{\rho}(t, q, \phi) \, dq \, d\phi
\tag{B.13}
$$

Substituting in the Fokker-Planck equation (B.3) for $\dot{\rho}$, we obtain

$$
\begin{aligned}
\dot{S}_{tot} &= \iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left\{ \frac{\partial}{\partial q} \left[\left(h(\phi) + \frac{f(q)}{R(t)} \right) \rho(t, q, \phi) \right] - f(q) \frac{\partial}{\partial \phi} \left[\rho(t, q, \phi) \right] \right. \\
&\qquad\qquad \left. + \frac{\partial^2}{\partial q^2} \left[\frac{kT}{R(t)} \, \rho(t, q, \phi) \right] \right\} \, dq \, d\phi \\
&= \iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left[h(\phi) \frac{\partial \rho(t, q, \phi)}{\partial q} - f(q) \frac{\partial \rho(t, q, \phi)}{\partial \phi} \right] dq \, d\phi \\
&\quad + \iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \frac{\partial}{\partial q} \left[\frac{f(q)}{R(t)} \rho(t, q, \phi) + \frac{kT}{R(t)} \, \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi
\end{aligned}
\tag{B.14}
$$

Let us perform integration by parts on the terms of the next-to-last line. The "*uv*"

term ($\int u \, dv = uv - \int v \, du$) vanishes because ρ vanishes exponentially fast away from the origin in q or ϕ. One term will be integrated by parts with respect to q, the other by ϕ.

$$\iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left[h(\phi) \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi$$

$$= -\iint h(\phi) \, \rho(t, q, \phi) \frac{\partial}{\partial q} \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] dq \, d\phi$$

$$= -\iint h(\phi) \, \rho(t, q, \phi) \left[-k \frac{1}{\rho} \frac{\partial \rho}{\partial q} - \frac{1}{T} \, f(q) \right] dq \, d\phi$$

$$\iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left[-f(q) \frac{\partial \rho(t, q, \phi)}{\partial \phi} \right] dq \, d\phi$$

$$= \iint f(q) \, \rho(t, q, \phi) \frac{\partial}{\partial \phi} \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] dq \, d\phi$$

$$= \iint f(q) \, \rho(t, q, \phi) \left[-k \frac{1}{\rho} \frac{\partial \rho}{\partial \phi} - \frac{1}{T} \, h(\phi) \right] dq \, d\phi$$

Then, combining these two results and canceling a common term, the next-to-last line of (B.14) is

$$\iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left[h(\phi) \frac{\partial \rho(t, q, \phi)}{\partial q} - f(q) \frac{\partial \rho(t, q, \phi)}{\partial \phi} \right] dq \, d\phi$$

$$= \iint k \left[h(\phi) \frac{\partial \rho}{\partial q} - f(q) \frac{\partial \rho}{\partial \phi} \right] dq \, d\phi$$

This integral can be computed. Because of the exponential decay of ρ,

$$\int h(\phi) \frac{\partial \rho}{\partial q} \, dq = h(\phi) \, \rho(t, q, \phi) \Big|_{q=-\infty}^{q=\infty} = 0,$$

and the definite integral of 0 with respect to ϕ is still 0. Similarly,

$$\int f(q) \frac{\partial \rho}{\partial \phi} \, d\phi = 0$$

The next-to-last line of (B.14) contained only drift terms, which are noiseless, it should not be surprising that it vanished. Also note that the resistance did not appear in these terms, so it did not matter that $R(t)$ was time-varying.

The last line of (B.14) will be integrated by parts with respect to q.

$$
\begin{aligned}
\dot{S}_{tot} &= \iint \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \frac{\partial}{\partial q} \left[\frac{f(q)}{R(t)} \rho(t, q, \phi) + \frac{kT}{R(t)} \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi \\
&= -\iint \frac{\partial}{\partial q} \left[-k \ln \rho - \frac{1}{T} E_{LC} \right] \left[\frac{f(q)}{R(t)} \rho(t, q, \phi) + \frac{kT}{R(t)} \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi \\
&= \iint \left[k \frac{1}{\rho} \frac{\partial \rho(t, q, \phi)}{\partial q} + \frac{1}{T} f(q) \right] \left[\frac{f(q)}{R(t)} \rho(t, q, \phi) + \frac{kT}{R(t)} \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi \\
&= \iint \frac{k}{\rho(t, q, \phi)} \left[\frac{\partial \rho}{\partial q} + \frac{\rho}{kT} f(q) \right] \frac{kT}{R(t)} \left[\frac{f(q)}{kT} \rho(t, q, \phi) + \frac{\partial \rho(t, q, \phi)}{\partial q} \right] dq \, d\phi \\
&= \frac{k^2 T}{R(t)} \iint \frac{1}{\rho(t, q, \phi)} \left[\frac{f(q) \, \rho(t, q, \phi)}{kT} + \frac{\partial \rho(t, q, \phi)}{\partial q} \right]^2 dq \, d\phi \geq 0
\end{aligned}
$$

The entropy is therefore increasing monotonically, showing that Second Law of Thermodynamics holds for an RLC circuit containing a time-varying R and nonlinear L and C. Again, it can be seen from the equations for $\dot{S}_{LC}$ (B.10) and $\dot{S}_R$ (B.12) that for an equilibrium density, $\dot{\rho} = 0$, the entropy change is identically zero. This can also be seen in the last line of the last equation, using (B.6) for $\partial \rho / \partial q$.

Notice that the time-varying $R(t)$ did not affect any of the calculations. It does not matter whether $R(t)$ is varying deterministically (controlled by some other signal, such as a MOSFET gate voltage) or stochastically ($1/f$ noise).

Appendix C

Existence and Uniqueness of Solutions to the Forward Equation

This Appendix considers the question: when do solutions exist, and are they unique, for differential equations driven by jump processes whose rates depend on the state of the system. Most of the literature on differential equations driven by jump processes concerns Poisson counters that have deterministic rates, though the rates may be time-varying.

So-called "doubly-stochastic" Poisson processes have stochastic rates [62, 63]. However, the rate is generally fully determined and then the point process is constructed according to the rate. In our case, we want to allow the rate to depend on the past of the point process. Therefore, we really have two questions:

(i) Does there exist a stochastic process with the (stochastic, state-dependent) rate we want?

(ii) Does the forward equation driven by this process have a solution that is unique?

The theorems of Brémaud [62] may provide some useful machinery. Under certain conditions on the stochastic process λ_t, which is chosen *ab initio* from a certain

distribution, there exists a counting process with a rate given by that stochastic process λ_t.

For this work, one complication is that λ_t depends on the counting process. However, λ_t only depends on the *past* of the counting process, so it is *measurable* with respect to the filtration or past of the counting process. A further complication is that the rate process λ_t is not bounded. For the diode model, as $v \to \infty$, $f_T(v)$ grows without bound. However, the steady-state probability is vanishing even more quickly as $v \to \infty$, and the function $f_T(v)$ describes the rate of transition to smaller voltage, so that we expect the rate process to satisfy

$$\int_0^t \lambda_s \, ds < \infty$$

almost surely for $t \geq 0$.

This machinery was used in [64] to model neuron activity by an "integrate and fire" method. For a system of equations like

$$\begin{aligned} dx &= -x \, dt + dy \\ dy &= f(x(t-)) \, dN \end{aligned}$$

where $f(\cdot) = 0$ or 1 and N is a homogeneous Poisson counter of rate μ, then $y(t)$ is a counting process with (state-dependent) rate $\mu f(x(t-))$. Generally, $f(x) = 0$ for x less than a threshold: the neuron's internal cell potential must cross this threshold, and then it will fire after some random time determined by the homogeneous Poisson process.

We have not completely understood these results, nor do we understand the relation between existence of these counting processes and existence of solutions to the forward equations.

In the case of a circuit consisting of a diode and a linear capacitor, we can say

more about the existence to solutions of the forward equation (in this case, a Master Equation) for the Poisson model description of the circuit. We can understand the discrete state space system of the Poisson model as a doubly-infinite discrete-state Markov process. Since the Poisson sources only supply single electrons, the capacitor charge is always an integer. The states of the Markov processes correspond to numbers of electrons on the capacitor's upper plate, and the transition rates out of each state are defined by the complicated, but predetermined, rates f_n and r_n.

Existence and uniqueness of the equilibrium solution, as mentioned in Chapter 5, is determined by the criterion of Gallager [13]: If there exists a set of non-negative numbers $\{p_n\}$ such that

(i) $\sum p_n = 1$,

(ii) $p_n \, \nu_n = p_{n+1} \, f_{n+1} + p_{n-1} \, r_{n-1}$, and

(iii) $\sum p_n \nu_n < \infty$,

where $\nu_n = f_n + r_n$ is the total rate of transitions out of the state n, then $\{p_n\}$ is the unique steady-state distribution.

We insist that the equilibrium distribution be Gibbs,

$$p_n^o = A \exp \left(\frac{-n^2 e^2}{2CkT} \right),$$

where A normalizes the distribution to satisfy (i). Not only does this distribution satisfy (ii), but it also satisfies detailed balance,

$$p_n^o \, r_n = p_{n+1}^o \, f_{n+1}.$$

In Chapter 5, it was shown that (iii) is satisfied for the diode model.

Recall the following from Chapter 2:

$$r_n = r_T\left(\frac{(n+1/2)e}{C}\right)$$

$$f_n = f_T\left(\frac{(n-1/2)e}{C}\right)$$

$$\frac{f_{n+1}}{r_n} = \exp\left(\frac{(n+1/2)e}{Cv_T}\right)$$

$$g(v) = e\left[f_T(v) - r_T(v)\right].$$

These equations can be solved for

$$f_n = \frac{g\left(\frac{(n-1/2)e}{C}\right)\exp\left(\frac{(n-1/2)e}{Cv_T}\right)}{\exp\left(\frac{(n-1/2)e}{Cv_T}\right) - 1}$$

$$r_n = \frac{g\left(\frac{(n+1/2)e}{C}\right)}{\exp\left(\frac{(n+1/2)e}{Cv_T}\right) - 1}$$

One needs very lax conditions on $g(v)$ for the summation $\sum p_n v_n$ to be bounded such that the equilibrium density is valid. In particular, any exponential constitutive relation will likely be dominated by the exponential of n^2 in the Gibbs distribution.

Next, we must ask about the existence (and uniqueness) of solutions for other initial conditions. Let us specifically consider the diode connected to a linear capacitor, because this example is sufficient to show the difficulties. Recall that this system had an equivalent Markov chain description with the forward equation

$$\dot{p}_n = \frac{I_S}{e}\exp\left(\frac{(n+1/2)e^2}{CkT}\right)p_{n+1} - \frac{I_S}{e}\left[1 + \exp\left(\frac{(n-1/2)e^2}{CkT}\right)\right]p_n + \frac{I_S}{e}\,p_{n-1}.$$

For the purposes of this section, we may rescale time to incorporate the constant I_S/e. Further, by properties of the exponential,

$$\exp\left(\frac{(n+1/2)e^2}{CkT}\right) = \left[\exp\left(\frac{e^2}{CkT}\right)\right]^{n+1/2}$$

which allows us to incorporate all the physical constants into the quantity

$$a \triangleq \exp\left(\frac{e^2}{CkT}\right). \tag{C.1}$$

Comparing this to the Gibbs distribution expression, $a = \exp(1/\sigma^2)$, where σ would be the standard deviation of a continuous Gaussian distribution. (Numerically, it appears also to be the standard deviation of the discrete Gibbs distribution, but I cannot prove this analytically.)

Therefore, our forward equation is mathematically equivalent to the expression

$$\dot{p}_n = a^{n+1/2}\, p_{n+1} - \left[1 + a^{n-1/2}\right] p_n + p_{n-1}. \tag{C.2}$$

Megretski suggested a clever way to find eigenvalues of this system. For the set of distributions

$$\left\{ p_n : \sum_n a^{n^2/2}\, p_n^2 < \infty \right\} \tag{C.3}$$

he shows that the spectrum is real and that an infinite number of eigenvalues can be obtained by the z-transform [65]. (There may be other eigenvalues that are not obtained by this method, but it turns out that the eigenvalues we want for the next appendix are obtained from the z-transform.) However, we will also see that this condition is not sufficient to guarantee that a solution exists for some initial distributions.

Note for any initial distribution $\{p_n(0)\}$ satisfying this condition, if a solution exists, then $F(t) \triangleq \sum_n a^{n^2/2}\, p_n^2(t)$ is monotonically non-increasing in time along the trajectory of the solution.

$$
\begin{aligned}
\frac{d}{dt}F(t) &= \frac{d}{dt}\sum_n a^{n^2/2}\, p_n^2 \;=\; \sum_n a^{n^2/2}\, 2\, p_n\, \dot{p}_n \\
&= \sum_n a^{n^2/2}\, 2\, p_n \left(a^{n+1/2}\, p_{n+1} - \left[1 + a^{n-1/2}\right] p_n + p_{n-1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_n a^{n^2/2}\, 2\, p_n \left(a^{n+1/2}\, p_{n+1} - p_n\right) + \sum_n a^{n^2/2}\, 2\, p_n \left(-a^{n-1/2}\, p_n + p_{n-1}\right) \\
&= \sum_n a^{n^2/2}\, 2\, p_n \left(a^{n+1/2}\, p_{n+1} - p_n\right) + \sum_n a^{(n+1)^2/2}\, 2\, p_{n+1} \left(-a^{n+1/2}\, p_{n+1} + p_n\right) \\
&= \sum_n 2\, a^{n^2/2} \left(p_n - a^{n+1/2}\, p_{n+1}\right) \left(a^{n+1/2}\, p_{n+1} - p_n\right) \\
&= -\sum_n 2\, a^{n^2/2} \left(p_n - a^{n+1/2}\, p_{n+1}\right)^2 \\
&\le\ 0
\end{aligned}
\tag{C.4}
$$

In fact, $F(t)$ is strictly decreasing unless $p_n = a^{n+1/2}\, p_{n+1}$ for all n, which is precisely the condition for the Gibbs equilibrium distribution.

Let us proceed to find the eigenvalues. The z-transform of a sequence $p_n(t)$ is given by

$$
p(z,t) \triangleq \sum_n z^{-n}\, p_n(t).
$$

We may transform the forward equation (C.2) to obtain

$$
\begin{aligned}
\dot{p}(z,t) &= \frac{z}{\sqrt{a}}\, p(z/a,t) - p(z,t) - \frac{1}{\sqrt{a}}\, p(z/a,t) + \frac{1}{z}\, p(z,t) \\[2mm]
&= \frac{z-1}{\sqrt{a}}\, p(z/a,t) - \frac{z-1}{z}\, p(z,t)
\end{aligned}
\tag{C.5}
$$

Now, we may consider evaluating this expression for certain values of z and solving the resulting differential equations. For $z = 1$, clearly $\dot{p}(1,t) = 0$, either from Eq. (C.5) or directly because $p(1,t) = \sum_n p_n(t) = 1$ for all t. Now, consider powers of a.

$$
\begin{aligned}
\dot{p}(a,t) &= \frac{a-1}{\sqrt{a}}\, p(1,t) - \frac{a-1}{a}\, p(a,t) \\[2mm]
\dot{p}(a^2,t) &= \frac{a^2-1}{\sqrt{a}}\, p(a,t) - \frac{a^2-1}{a^2}\, p(a^2,t) \\
&\qquad \cdots \\[1mm]
\dot{p}(a^n,t) &= \frac{a^n-1}{\sqrt{a}}\, p(a^{n-1},t) - \frac{a^n-1}{a^n}\, p(a^n,t)
\end{aligned}
$$

Clearly, the eigenvalues of this system are

$$\tilde{\lambda}_n = -\frac{a^n - 1}{a^n} = -1 + a^{-n}, \quad n = 0, 1, 2, \ldots \tag{C.6}$$

These will be used in the next appendix to find the power spectral density of the diode-capacitor system.

However, now let us consider negative powers of a. We re-write Eq. (C.5) to read

$$\dot{p}(z, t) = \frac{z - 1}{\sqrt{a}} \, p(z/a, t) + \left(\frac{1}{z} - 1 \right) \, p(z, t).$$

Then,

$$
\begin{aligned}
\dot{p}(1/a, t) &= \frac{1/a - 1}{\sqrt{a}} \, p(1/a^2, t) + (a - 1) \, p(1/a, t) \\
\dot{p}(1/a^2, t) &= \frac{1/a^2 - 1}{\sqrt{a}} \, p(1/a^3, t) + (a^2 - 1) \, p(1/a^2, t) \\
&\cdots \\
\dot{p}(1/a^n, t) &= \frac{1/a^n - 1}{\sqrt{a}} \, p(1/a^{n+1}, t) + (a^n - 1) \, p(1/a^n, t)
\end{aligned}
$$

Suppose there were an initial distribution $\{p_n(0)\}$ such that its z-transform satisfied the following:

$$
\begin{aligned}
p(1/a, 0) &\neq 0 \\
p(1/a^n, 0) &= 0, \quad n = 2, 3, \ldots
\end{aligned}
$$

The initial conditions for $p(1/a^n, t)$, along with their differential equations, mean that

$$p(1/a^n, t) \equiv 0, \quad n = 2, 3, \ldots$$

but

$$p(1/a, t) = \exp\left[(a - 1)t\right] p(1/a, 0).$$

Since $a > 1$, this shows that $p(1/a, t)$ grows exponentially, even though $F(t) = \sum_n a^{n^2/2} \, p_n^2(t)$ is monotonically non-increasing along trajectories.

The initial distribution with z-transform

$$p(z, 0) = \prod_{m=2}^{\infty} \left(1 - \frac{1}{az} \right)$$

has the initial conditions required.

Megretski [65] concludes that there is no solution to the system of equations (C.2) for this initial distribution, so we need a stronger condition on the initial distribution than that expressed by (C.3). We believe that any distribution with a finite number of non-zero p_n will be well-behaved, as will, of course, any equilibrium density.

Appendix D

Poisson Model Power Spectral Density

In this appendix, we will consider under what circumstances the Poisson model for shot noise will differ appreciably from the Gaussian model applied to a linearized conductance. We will investigate this by means of the power spectral density of a diode connected to a capacitor. In any measurement technique for the power spectral density, there will be a capacitance in the circuit: the input capacitance of the instrument, if not others.

D.1   Symbolic Analysis

The power spectral density of the linearized Gaussian model will have a lowpass characteristic of an RC filter. For a nonlinear device of constitutive relation $i = g(v)$, the incremental conductance at the origin is $G = g'(v)|_{v=0}$. The critical frequency is then $w_{\text{crit}} = G/C$. For the diode,

$$G = g'(v)|_{v=0} = \frac{d}{dv} I_S \left[\exp(v/v_T) - 1 \right] \Big|_{v=0} = \frac{I_S}{v_T},$$

so that the critical frequency is $w_{\mathrm{crit}} = I_S/(Cv_T)$. The power spectral density looks like the top-most curve in Fig. D-1.

For the Poisson model, the analysis is more difficult. We will start with the Markov chain of Fig. 2-8. We will attempt to find the autocorrelation of this chain, which will be expressed in the form of a sum of exponentials. The Fourier transform of the autocorrelation is the power spectral density. If one of the exponentials in the autocorrelation is dominant, then we expect the Fourier transform to have a critical frequency corresponding to this eigenvalue.

The forward equation for the Markov chain is

$$\dot{p}_n = \frac{I_S}{e} \exp\left(\frac{(n+1/2)e^2}{CkT}\right) p_{n+1} - \frac{I_S}{e}\left[1 + \exp\left(\frac{(n-1/2)e^2}{CkT}\right)\right] p_n + \frac{I_S}{e}\, p_{n-1}.$$

As in the previous appendix, we will rescale time by I_S/e and use the variable $a = \exp\left(e^2/CkT\right)$ to absorb the remaining physical constants. We will be interested in the behavior of this system in the limits that $e^2/(CkT)$ is either large or small. In the case that this quantity is small (since e and k are fundamental physical constants, this means CT large), we expect to approach the linearized Gaussian result, based on our analysis of the Poisson model when the electron size goes to zero. In the case that this quantity is large, we hope to find a measurable difference in the power spectral densities predicted by the Poisson and Gaussian models.

Hiding these physical constants yielded the equation (C.2), which can be expressed in matrix form,

$$\dot{\mathbf{p}} = \mathbf{Q}\,\mathbf{p}$$

where $\mathbf{p} = [\ldots, p_2, p_1, p_0, p_{-1}, \ldots]^T$ and

$$
\mathbf{Q} = \begin{bmatrix}
\ddots & \ddots & 0 & 0 & 0 \\
a^{3/2} & -[1 + a^{1/2}] & 1 & 0 & 0 \\
0 & a^{1/2} & -[1 + a^{-1/2}] & 1 & 0 \\
0 & 0 & a^{-1/2} & -[1 + a^{-3/2}] & 1 \\
0 & 0 & 0 & \ddots & \ddots
\end{bmatrix}
\tag{D.1}
$$

The Gibbs equilibrium distribution is

$$
\mathbf{p}^o = A \left[\ldots, a^{-2^2/2}, a^{-1^2/2}, a^{-0^2/2}, a^{-1^2/2}, \ldots \right]^T = A \left[\ldots, a^{-2}, a^{-1/2}, 1, a^{-1/2}, \ldots \right]^T,
$$

where $A = \left(\sum_{n=-\infty}^{\infty} a^{-n^2/2} \right)^{-1}$ normalizes the distribution. This solution satisfies Gallager's criterion for uniqueness of a steady-state distribution. We believe that solutions to the forward equation will exist for any initial distribution of compact support, specifically any delta-function corresponding to the exact charge on the capacitor when we set up the experiment.

The eigenvalues

$$
\tilde{\lambda}_n = -\frac{a^n - 1}{a^n} = -1 + a^{-n}, \quad n = 0, 1, 2, \ldots,
\tag{D.2}
$$

were calculated in the previous appendix. The eigenvalue for $n = 0$ is zero; this corresponds to the equilibrium distribution eigenvector p_n^o. Recalling that we rescaled time by I_S/e, the next eigenvalue of the original system is

$$
\lambda_1 = \frac{I_S}{e} \left[-1 + a^{-1} \right] = \frac{I_S}{e} \left[-1 + \exp\left(-\frac{e^2}{CkT} \right) \right]
$$

In the limit that CT is large (which means the approximate variance $CkT/e^2 = \sigma^2$

is large), the exponential can be Taylor-expanded

$$\lambda_1 \approx \frac{I_S}{e} \left[-1 + 1 - \frac{e^2}{CkT} \right] = -\frac{I_S}{C\, v_T}, \tag{D.3}$$

which is the critical frequency obtained in the linearized Gaussian approximation. To see specifically how this eigenvalue appears in the power spectral density for the Poisson system, we will have to make some more calculations.

The autocorrelation of the charge on the top plate of the capacitor is given by the expression

$$R_{qq}(t) = E\left\{ q(t)\, q(0) \right\}$$

(The system is assumed to be at equilibrium, so that the random process is stationary: $E\left\{ q(t)\, q(0) \right\} = E\left\{ q(t+\tau)\, q(\tau) \right\}$ for any τ.) Since we have a discrete-state system, we can split up the expectation as follows:

$$\begin{aligned}
R_{qq}(t) &= \sum_n E\left[q(t)\, q(0)\, |q(0) = n \right] \Pr\{q(0) = n\} \\
&= \sum_n nE\left[q(t)\, |q(0) = n \right] \Pr\{q(0) = n\} \\
&= \sum_n n \left[\sum_m m \Pr\{q(t) = m\, |q(0) = n\} \right] \Pr\{q(0) = n\} \qquad \text{(D.4)}
\end{aligned}$$

If we define

$$P_{ij}(t) \overset{\Delta}{=} \Pr\{q(t) = j\, |q(0) = i\},$$

then the Kolmogorov equations in [13, Sec. 6.2],

$$\mathbf{P}(t) = \sum_{i=0}^{\infty} \frac{t^i \mathbf{Q}^i}{i!} = \exp\left(\mathbf{Q}t \right),$$

give us a way to calculate the transition probabilities $P_{nm}(t)$. Here, $\mathbf{P}(t)$ is the matrix with elements $P_{nm}(t)$, $\mathbf{P}(0) = \mathbf{I}$, and $\mathbf{Q}$ is the same matrix we defined in Eq. (D.1) for the forward equation.

If we define $\mathbf{c} = [\ldots, -2, -1, 0, 1, \ldots]$, we can write (D.4) in matrix form,

$$R_{qq}(t) = \mathbf{c}^T \, \mathbf{P}(t) \, [\text{diag } \mathbf{p}^\text{o}] \, \mathbf{c} = \mathbf{c}^{\mathrm{T}} \, \exp{(\mathbf{Q}t)} \, [\text{diag } \mathbf{p}^\text{o}] \, \mathbf{c}, \qquad (D.5)$$

where $[\text{diag } \mathbf{p}^\text{o}]$ is the matrix with the vector $\mathbf{p}^o$ along the main diagonal and zeros elsewhere.

It can be shown [65] that the system matrix $\mathbf{Q}$ is similar to a symmetric matrix. Let $\mathbf{D}$ be the diagonal matrix with entries $d_{nn} = a^{n^2/4}$ where $a = \exp(e^2/CkT)$ was defined earlier. Then

$$\mathbf{D}\,\mathbf{Q}\,\mathbf{D}^{-1} = -\mathbf{D}\,\mathbf{M}\,\mathbf{D} = \begin{bmatrix} \ddots & -a^{3/4} & 0 & 0 & 0 \\ -a^{3/4} & 1+a^{1/2} & -a^{1/4} & 0 & 0 \\ 0 & -a^{1/4} & 1+a^{-1/2} & a^{-1/4} & 0 \\ 0 & 0 & a^{-1/4} & 1+a^{-3/2} & -a^{-3/4} \\ 0 & 0 & 0 & -a^{-3/4} & \ddots \end{bmatrix}$$

where $\mathbf{M}$ is positive semidefinite and symmetric with a nullspace of dimension 1 consisting of the span of the vector $\mathbf{1}$. This can be shown algebraically or by reversibility, noting that $p_n^o \propto a^{-n^2/2}$. In this case, the symmetric form for the autocorrelation is

$$R_{qq}(t) = \mathbf{c}^T \, \mathbf{D}^{-1} \exp{[-\mathbf{D}\,\mathbf{M}\,\mathbf{D}\,t]}\,\mathbf{D}^{-1}\mathbf{c} \qquad (D.6)$$

In either case, Eq. (D.5) or (D.6), we would like to show that the scalar time function R_{qq} is a sum of scaled exponentials, $b_0 \exp(\lambda_0 t) + b_1 \exp(\lambda_1 t) + \ldots$, with the eigenvalues determined in Eq. (D.2). For large CT, we expect to find that one of the coefficients, specifically b_1, is dominant, because the corresponding eigenvalue λ_1 matches the critical frequency from the linearized Gaussian case. In this case, the vector $\mathbf{D}^{-1}\mathbf{c}$ would be an eigenvector of the matrix $\mathbf{D}\,\mathbf{M}\,\mathbf{D}$ (and hence the matrix exponential).

Unfortunately, $\mathbf{D}^{-1}\mathbf{c}$ is never exactly an eigenvector for finite CT. Consider the center row of the eigenvalue equation:

$$\mathbf{D\,M\,D\,D}^{-1}\mathbf{c} \overset{?}{=} \lambda_1\,\mathbf{D}^{-1}\mathbf{c}$$

$$\begin{bmatrix} \cdots & 0 & -a^{1/4} & 1+a^{-1/2} & -a^{-1/4} & 0 & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ 2a^{-1} \\ a^{-1/4} \\ 0 \\ -a^{-1/4} \\ -2a^{-1} \\ \vdots \end{bmatrix} \overset{?}{=} \left(-1+a^{-1}\right)[0]$$

$$-1 + 0 + a^{-1/2} \overset{?}{=} 0$$

This equation is only true when $a = 1$. For the eigenvalue, we used the constant and linear terms in the expansion for a; for the eigenvector, we only want the constant term.

It is not clear how to proceed analytically. Therefore, we will look for an expression for the power spectral density for the Poisson model and compare it to the lowpass filter function of the linearized Gaussian model. Fortunately, it is possible to find an analytical expression for the power spectral density, rather than having to numerically Fourier-transform the autocorrelation.

The power spectral density of the capacitor charge is the Fourier transform of the autocorrelation function.

$$S_{qq}(\omega) = \mathcal{F}\left\{R_{qq}(t)\right\} = \int_{-\infty}^{+\infty} e^{-j\omega t} R_{qq}(t)\,dt$$

The autocorrelation expression (D.5) is only valid for $t > 0$, but since the random

process is reversible, this is not an obstacle. For positive time,

$$
\int_0^{+\infty} e^{-j\omega t} R_{qq}(t)\ dt \ = \ \int_0^{+\infty} e^{-j\omega t}\ \mathbf{c}^T\ \exp\left(\mathbf{Q}t\right)\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c}\ dt
$$

$$
= \ \mathbf{c}^T \int_0^{+\infty} \exp\left(-j\omega t\mathbf{I} + \mathbf{Q}t\right)\ dt\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c}
$$

$$
= \ \mathbf{c}^T\ (j\omega\mathbf{I} - \mathbf{Q})^{-1}\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c}.
$$

The power spectral density is then twice the real part of the above expression, or the sum of that expression and its complex conjugate.

$$
S_{qq}(\omega) \ = \ \mathbf{c}^T\ (j\omega\mathbf{I} - \mathbf{Q})^{-1}\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c} + \mathbf{c}^T\ (-j\omega\mathbf{I} - \mathbf{Q})^{-1}\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c}
$$

$$
= \ \mathbf{c}^T\ \left[(j\omega\mathbf{I} - \mathbf{Q})^{-1} + (-j\omega\mathbf{I} - \mathbf{Q})^{-1}\right]\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c}
$$

$$
= \ -2\ \mathbf{c}^T\ \mathbf{Q}\ \left(\omega^2\mathbf{I} + \mathbf{Q}^2\right)^{-1}\ [\operatorname{diag}\mathbf{p}^{\mathrm{o}}]\ \mathbf{c} \tag{D.7}
$$

D.2   Numerical Analysis

The power spectral density (D.7) can be calculated with MATLAB. We consider a system with only a finite number of states, specifically $2N+1$ states, corresponding to $-N$ to N electrons on the capacitor. Of course, the equilibrium density will decrease exponentially with $|N|$, making computer calculations difficult. However, the effects of these distant states on the power spectral density will also be negligible. Some roundoff errors are avoided by using pico-units, such as pF of capacitance and pC for the charge of the electron. The constants always appear in ratios in the equations so that the scaling cancels algebraically.

If one simply sets the rates $f_{-N} = 0$ and $r_N = 0$, meaning no transition down from $-N$ to $-(N + 1)$ and no transition up from N to $N + 1$, then the states outside $-N \ldots N$ will never be reached. Fortuitously, the equilibrium density for the reachable states will be unchanged except for normalization. Detailed balance must still be satisfied for the reachable states, and is governed by rates that we have not

changed.

While it is possible to use detailed balance to calculate the equilibrium distribution, it is numerically more robust to simply use the Gibbs formula,

```
rho = exp(-[-N:N].^2 / (2*C*V_T))';
```

and then normalize. If the vectors `f` and `r` contain the transition rates for the states, then the tridiagonal matrix $\mathbf{Q}$ is given by

```
Q = diag(-f-r,0) + diag(r(2:2*N+1),1) + diag(f(1:2*N),-1);
```

The MATLAB expression for the power spectral density is

```
Sxx(i) = -2 * (E/C)^2 * c' * Q * inv(w(i)^2*eye(2*N+1)+Q^2)
                * diag(rho) * c;
```

The factor $(E/C)^2$ is the electron size over the capacitance, and converts the power spectral density in charge to one in voltage. This result is compared to the linearized Gaussian model.

```
Svv = 2 * k * T * G ./ (G^2 + C^2*w.^2);
```

where $G = I_S/v_T$ is the linearized conductance at the origin. The complete MATLAB codes are in Section D.4.

D.3   Numerical Results

In this section, we present some simple results. For the product $CT = 10$, MATLAB calculated the following:

```
sigma=  73.42        G/C = 1.159420e+03
N= 50  1:-2.504115e+07 2N:-6.643783e+03  2N+1:2.592060e-10
                       98.68%             0.00%
N=100  1:-2.515784e+07 2N:-2.176808e+03  2N+1:-1.392664e-09
                       99.04%             0.00%
N=296  1:-2.562321e+07 2N:-1.160312e+03  2N+1:-1.629009e-09
                       99.99%             0.00%
```

The first line gives the standard deviation $\texttt{sigma} = \sqrt{CkT/e^2}$ followed by the critical frequency G/C for the Gaussian model for the linearized system. We expect that the number of electrons N must be greater than the standard deviation of the charge on the capacitor plates to capture the behavior of the system.

There are three pairs of lines giving the data for $N = 50$, 100, and 296. The first line of each pair lists three of the $2N + 1$ eigenvalues of the truncated system. The largest, labelled 2N+1, is approximately zero, matching Eq. (D.2) for $n = 0$. The next-largest eigenvalue, labelled 2N, appears to be converging to the critical frequency G/C. The second line of each pair shows part of the decomposition of the vector $\mathbf{D}^{-1} \mathbf{c}$ in Eq. (D.6) in terms of the eigenvectors corresponding to the two largest eigenvalues. If we write $\mathbf{D}^{-1} \mathbf{c} = \sum_{n=1}^{2N+1} b_n \mathbf{v}_n$, where $\mathbf{v}_n$ are the eigenvectors, then the percentages given are $|b_k|^2 / \sum_{n=1}^{2N+1} |b_n|^2$ for $k = 2N, 2N + 1$. The percent of energy in the eigenvector corresponding to the eigenvalue that converges to the critical frequency G/C is almost 100%.

In Fig. D-1, the line for $N = 50$ is the lowest; $N = 100$ is the middle, and $N = 296$ lies almost exactly on top of the dotted curve of the linearized Gaussian lowpass filter curve. The value $N = 296$ was chosen as four times the standard deviation; 99.99% of the area under a Gaussian curve lies within four standard deviations.
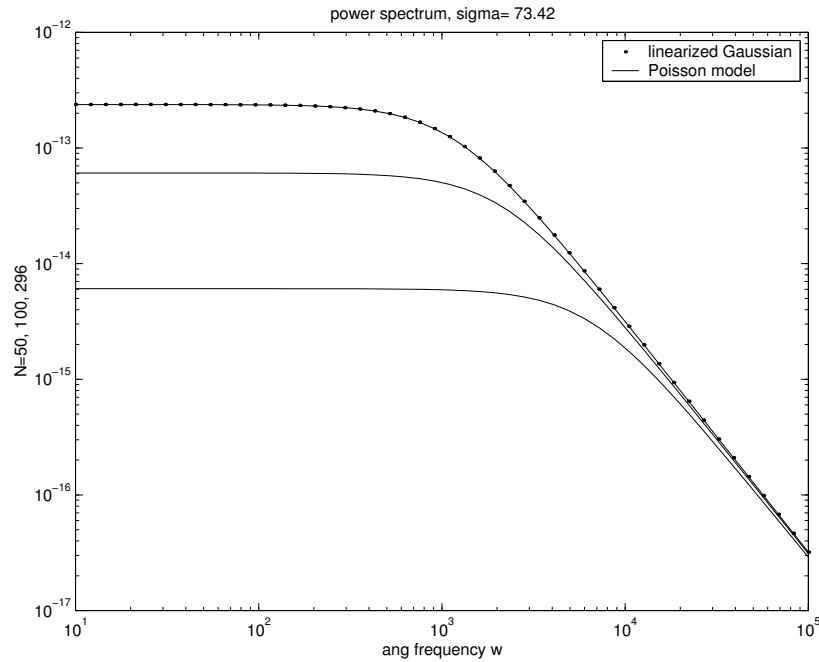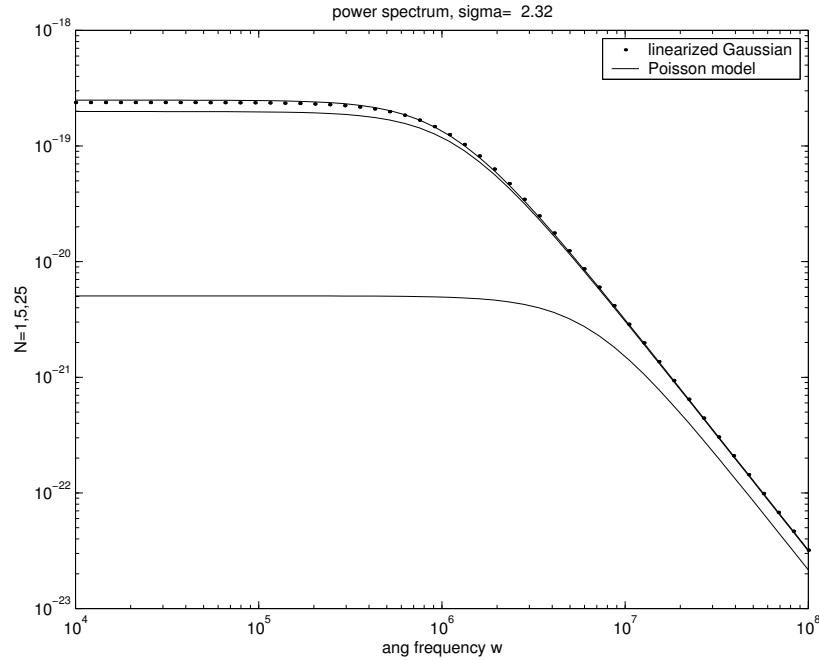
Figure D-1: Power spectral densities for $CT = 10$.

For large CT, *i.e.*, large `sigma`, we need a larger N to capture a larger section of the Markov chain. But, as predicted by our analysis on the eigenvalues and eigenvectors, as well as by physical reasoning, the more electrons (in a mean-square sense) are on the capacitor, the less influence a single electron will have on the behavior. The hope is that for small CT, the effects of a single charge would be more noticeable, and hence, the power spectral density for the Poisson model would differ appreciably from that predicted by the linearized Gaussian model.

For the case $CT = 0.01$, MATLAB computed

```
sigma =   2.32         G/C = 1.159420e+06
N=  1  1:-1.852184e+07 2N:-6.531965e+06  2N+1:4.656613e-10
                       99.82%            0.00%
N=  5  1:-3.102086e+07 2N:-1.203525e+06  2N+1:-9.313226e-10
                       96.31%            0.00%
N= 25  1:-6.215398e+08 2N:-1.058232e+06  2N+1:-1.874696e-10
                       91.01%            0.00%
```

Figure D-2: Power spectral densities for $CT = 0.01$.

Again, as N increases, the second-largest eigenvalue approaches the the critical frequency G/C. The curves for $N = 1$, 5, and 25 are plotted in Fig. D-2; again, the larger values of N are closer to the linearized Gaussian curve. However, note that for $N = 25$, the Poisson model curve is slightly above the linearized Gaussian curve. This effect persists for larger values of N (but recall that for more than 99% of the time, the system is certainly within the ten standard deviations represented by $N = 25$). Further analysis shows that when decomposing the vector $\mathbf{D}^{-1}\mathbf{c}$ into the eigenvectors of the symmetrized system, 91% is in the direction associated with the second-largest eigenvalue. A further 7%, most of the remainder, is in the direction associated with the third-largest eigenvalue, $\lambda_2 = -1 + a^{-2} \approx -1.937288\mathrm{e} + 06$, which is within a factor of 2 of the second-largest eigenvalue. (Note that any energy in the direction associated with the largest eigenvalue would be lost, because this eigenvalue is zero and hence makes no contribution to the power spectral density.)

An experimental disproof of the linearized Gaussian model would require an extremely sensitive spectrum analyzer to see the slightly higher response or extremely good frequency resolution to detect the second time-constant behavior. Of course, one also needs a diode that operates properly at low temperatures (where "carrier freeze-out" might occur) and for which one has very good knowledge of its saturation current to calculate the expected critical frequency.

D.4   MATLAB Code

shotcomp.m

```
% shotcomp.m                              % gjcoram, 3/21/00
% comparison of power spectral density for
% finite-state shot-noise model to linearized Gaussian

% parameters
%   in pico-units - note that I_S/E = pA/pC = A/C
%                   E/C = pC/pF, G/C = (pA/V)/pC
global I_S C E V_T MOD
I_S = 1;                          % saturation current, pA
if exist('C') == 0 C=10; end  % capacitance, pF
E=1.6e-7 ;                        % electron charge, pC
k=1.38e-11;                       % Boltzmann constant, pJ/K
if exist('T') == 0 T=10; end  % temperature, K
V_T = k*T/E;                      % thermal voltage, Volts
MOD = 1;                          % model for f and r: diode=1

% Gaussian linearized solution
%
% Svv = 2 kT G / (G^2 + C^2 w^2)
% corner freq: G/C = wc
G=I_S/V_T; %pA/V
wc=ceil(log10(G/C));
nfreq=50;
w=logspace(wc-3,wc+1,nfreq);
Svv = 2 * k * T * G ./ (G^2 + C^2*w.^2);
          %pJ  pA/V / ((pA/V)^2 + (pF/s)^2)
```

```
clf;
loglog(w,Svv,'k.');
title('power spectrum');
xlabel('ang frequency w');
hold on;

% std dev for eigenvalue comparison
sig2=C*k*T/E^2;
s=sprintf('sigma =%7.2f        G/C = %e',sqrt(sig2),G/C);
disp(s);
s=sprintf('power spectrum, sigma=%6.2f',sqrt(sig2));
title(s);

% shot-noise model solutions
%
if (sig2<10)
  for N=[1,5,25];
    Sxx=shotspec(N,w);
    loglog(w,Sxx,'y');
  end
  ylabel('N=1,5,25');
else
  four = 4*ceil(sqrt(sig2));
  for N=[50, 100, four];
  Sxx=shotspec(N,w);
  loglog(w,Sxx,'g');
  end
  s=sprintf('N=50, 100, %d',four);
  ylabel(s);
end

% make sure Gaussian is on top
loglog(w,Svv,'k.');
legend('linearized Gaussian','Poisson model');
```

shotspec.m

```
function Sxx=shotspec(N,w)
% function Sxx=shotspec(N,w)                    % gjcoram, 3/21/00
% power spectral density for finite-state shot-noise model
% chain runs from -N to N, frequency points w

% parameters
%
global I_S C E V_T MOD
c=[N:-1:-N]';

% get transition rates
%
% reverse (rising charge) rates
r=rev(N);
r(1)=0;      % no transition from N to N+1
%
% forward (falling charge) rates
f=forw(N);
f(2*N+1)=0;  % no transition from -N to -N-1



% eq dist from Maxwell-Boltzmann
%
rho = exp(-c.^2 * E / (2*C*V_T))';
% normalize
a=1/sum(rho);
rho=a*rho;

% find the transition matrix
%
% Q(i,i) = -nu_i = -(f_i + r_i)
% Q(i,j) = q_ij = P_ij*nu_i
% q_i,i+1 = r_i;  q_i,i-1 = f_i;  q_i,j=0, otherwise
%   Q(i,i)          Q(i,i+1)           Q(i,i-1)
Q = diag(-f-r,0) + diag(r(2:2*N+1),1) + diag(f(1:2*N),-1);

% symmetrize (so eigenvectors are orthogonal)
%
D=diag(sqrt(rho)); % actually inv(D) in Wyatt's report
P=inv(D)*Q*D;
```

```
% check eigenvalues - 2nd largest -> G/C ??
%
[v,l]=eig(P);
[eigvs,i]=sort(diag(l));
s=sprintf('N=%3d  1:%e 2N:%e  2N+1:%e',
          N,eigvs(1),eigvs(2*N),eigvs(2*N+1));
disp(s)

% check expansion in 2 largest eigenvectors
%
Dr=D*c;
an=v'*Dr;  %Dr = a1 v1 + a2 v2 + ...
pv0=an(i(2*N+1))^2/sum(an.^2); % fraction along 0-eigenvalue direction
pv1=an(i(2*N))^2/sum(an.^2);   % fraction along next largest ev
s=sprintf('                        %3.2f%%             %3.2f%%',
          100*pv1, 100*pv0);
disp(s)

% power spectral density from Report 1, eq. (4)
%
Q2=Q^2;
nfreq=size(w,2);
Sxx=zeros(1,nfreq);
for i=1:nfreq
  Sxx(i) = -2 * (E/C)^2 * c' * Q * inv(w(i)^2*eye(2*N+1)+Q2)
            * diag(rho) * c;
end
```

forw.m

```
function y=forw(N)
% computes forward rates
% N = number of states in chain
% model = 1 for diode

global I_S C E V_T MOD

if MOD==1
  % for diode, f_j=(I_S/e)*exp(-e/(2*C*V_T))*exp(j*e/(C*V_T))
  j=[N:-1:-N];
  y=I_S/E*exp(-E/(2*C*V_T))*exp(j*E/(C*V_T));
elseif MOD==2
  % diode w/half-electron mean?
  % for diode, f_j=(I_S/e)*exp(j*e/(C*V_T))
  j=[N:-1:-N];
  y=I_S/E*exp(j*E/(C*V_T));
else
  disp('error! - no such model')
end
```

rev.m

```
function y=rev(N)
% computes reverse rates
% N = number of states in chain
% model = 1 for diode

global I_S C E V_T MOD

% for diode, r_j=I_S/e
if MOD==1
  y=ones(1,2*N+1)*I_S/E;
elseif MOD==2
  y=ones(1,2*N+1)*I_S/E;
else
  disp('error! - no such model')
end
```

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