Local Time-Space Calculus and Extensions of Itô's Formula

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Abstract. Let $X = (X_t)_{t \geq 0}$ be a continuous semimartingale and let $F : I\!R_+ \times I\!R \to I\!R$ be a C^1 function. Then the change-of-variable formula is valid:

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(s, X_s)\, ds + \int_0^t F_x(s, X_s)\, dX_s \\ - \frac{1}{2} \int_0^t \int_{I\!R} F_x(s, x)\, d\ell_s^x$$

where ℓ_s^x is the local time of X at x defined by:

$$\ell_s^x = I\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s I(x \leq X_r < x + \varepsilon)\, d\langle X, X \rangle_r$$

and $d\ell_s^x$ refers to an area integration with respect to $(s, x) \mapsto \ell_s^x$. Further extensions of this formula for non-smooth functions F are also briefly examined. The approach leads to a formal $d\ell_t^x$ calculus which appears useful in guessing a candidate formula for $F(t, X_t)$ before a rigorous proof is known or given.

1. Introduction

The fundamental result of stochastic calculus is Itô's formula (2.1) firstly established by Itô [5] for a standard Brownian motion and then later extended to continuous semimartingales by Kunita and Watanabe [7]. [For simplicity in this article we will not consider semimartingales with jumps.] The function F appearing in Itô's formula is C^2 in the space variable, and the correction to the classic Leibnitz-Newton formula (the final term in (2.1) below) is expressed by means of the quadratic variation (2.2).

Various extensions of the Itô formula have been established for functions F which are not C^2 in the space variable. The best know of these extensions is the Itô-Tanaka formula (2.3) firstly derived by Tanaka [13] for $F(x) = |x|$ and then extended to absolutely continuous F with F' of bounded variation by Meyer [9] and Wang [14]. The correction term appearing in this formula is expressed by means of the local time (2.4) which goes back to Lévy [8] (see e.g. [6] or [12]).

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A different extension to absolutely continuous F with locally bounded F' due to Bouleau and Yor [1] is given in (2.5) below. The correction term appearing in this formula is also expressed by means of the local time (2.4), however, in a different manner which suggests a formal integration by parts. Both formulas (2.3) and (2.5) are derived only in dimension one.

Motivated by applications in free-boundary problems of optimal stopping [10] we have recently derived an extension of Itô's formula [11] stated in (2.11) below. The most interesting in this formula, in comparison with the extensions above, is its final term where the possible jump of $F_x(s,\cdot)$ along the curve $s \mapsto b(s)$ is integrated with respect to the *time variable* s of the local time ℓ_s^b from (2.12) below.

In our attempts to understand a more general rule unifying the various correction terms reviewed above, we have noticed that Eisenbaum [3] made a fundamental contribution in the case of standard Brownian motion by deriving the extension of Itô's formula stated in (2.8) below. The correction term in this formula is expressed as an area integral with respect to both the time variable s and the space variable x of the local time ℓ_s^x from (2.4) below. The arguments of Eisenbaum rely on combining the Bouleau-Yor extension (2.5) with the Föllmer-Protter-Shiryaev extension (2.6) and thus strongly depend on the time-reversal property of standard Brownian motion.

The main aim of the present article is to make use of the formula (2.11) below and show that the representation of the correction term in the Eisenbaum's extension (2.11) as an area integral with respect to the local time-space (as we call it in short) is not only restricted to a time-invariant Brownian motion process but extends quite generally to all continuous semimartingales. This is firstly done for C^1 functions F in Section 3, and then extended to absolutely continuous functions F with F_t and F_x of bounded variation in Section 4.

We make no attempt in the present article to specify the most general class of functions H for which the double integral $\int_0^t \int_{I\!R} H(s,x)\, d\ell_s^x$ makes sense. Instead in Section 5 we will show how a number of known extensions of the Itô formula can be obtained by formal manipulations of the $d\ell_s^x$ integral. This formalism (or formal $d\ell_t^x$ calculus as we call it) appears to be useful when one needs to guess a candidate formula for $F(t, X_t)$ before a rigorous proof is known or given. A typical example of this guessing mechanism is given in the end of Section 5. It remains a challenging task, however, to carry out this programme on firm mathematical grounds to a more satisfactory completion.

2. Itô formula and extensions

In this section we will review various extensions of the Itô formula for the purpose of comparison and for further reference.

1. **Itô formula** ([5], [7]). Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale (see e.g. [12]) and let $F : I\!R_+ \times I\!R \to I\!R$ be a $C^{1,2}$ function. Then we have:

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(s, X_s)\, ds + \int_0^t F_x(s, X_s)\, dX_s + \frac{1}{2} \int_0^t F_{xx}(s, X_s)\, d\langle X, X\rangle_s \tag{2.1}$$

where $\langle X, X\rangle_s$ is the quadratic variation of X given by:

$$\langle X, X\rangle_s = I\!P - \lim_{n \to \infty} \sum_{s_i \in D_s^n} (X_{s_i} - X_{s_{i-1}})^2 \tag{2.2}$$

and the set D_s^n consists of arbitrary points $0 = s_0 < s_1 < \ldots < s_{n-1} < s_n = s$ satisfying $\max_{1 \le i \le n}(s_i - s_{i-1}) \to 0$ as $n \to \infty$.

2. **Itô-Tanaka formula** ([13], [9], [14]). Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale and let $F : I\!R \to I\!R$ be an AC function with F' of BV . [Throughout AC stands for absolutely continuous, and BV for bounded variation. It is equivalent to $F = F_1 - F_2$ where F_1 and F_2 are convex functions.] Then we have:

$$F(X_t) = F(X_0) + \int_0^t F'_-(X_s)\, dX_s + \frac{1}{2} \int_{I\!R} \ell_t^x\, dF'(x) \tag{2.3}$$

where ℓ_t^x is the local time of X at the point x defined by:

$$\ell_t^x = I\!P\; - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t I(x \le X_s < x + \varepsilon)\, d\langle X, X\rangle_s. \tag{2.4}$$

The formula (2.3) remains valid if the left-derivative F'_- is replaced by the right-derivative F'_+ provided that $I(x \le X_s < x + \varepsilon)$ in (2.4) is replaced by $I(x - \varepsilon < X_s \le x)$.

3. **Bouleau-Yor extension** [1]. Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale and let $F : I\!R \to I\!R$ be an AC function with F' locally bounded. Then:

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\, dX_s - \frac{1}{2} \int_{I\!R} F'(x)\, d_x \ell_t^x \tag{2.5}$$

where $d_x \ell_t^x$ refers to an integration with respect to $x \mapsto \ell_t^x$. The latter function is known to be of unbounded variation generally, and so is the final term in (2.5) as a function of t .

4. **Föllmer-Protter-Shiryaev extension** [4]. Let $B = (B_t)_{t \ge 0}$ be a standard Brownian motion and let $F : I\!R_+ \times I\!R \to I\!R$ be a measurable function satisfying the following conditions: (i) $t \mapsto F(t, x)$ is AC ; (ii) $x \mapsto F(t, x)$ is AC; (iii) $x \mapsto F_t(t, x)$ belongs to L^1_{loc} ; (iv) $x \mapsto F_x(t, x)$ belongs to L^2_{loc} ; (v) $t \mapsto F_t(t, \cdot\,)$ is continuous from $I\!R_+$ to L^1_{loc} ; (vi) $t \mapsto F_t(t, \cdot\,)$ is continuous from $I\!R_+$ to L^2_{loc} . Then we have:

$$F(t, B_t) = F(0, B_0) + \int_0^t F_t(s, B_s)\, ds + \int_0^t F_x(s, B_s)\, dB_s + \frac{1}{2}[F_x(\,\cdot\,, B), B]_t \tag{2.6}$$

where $[F_x(\,\cdot\,,B),B]_t$ is the quadratic covariation given by:

$$(2.7)\quad [F_x(\,\cdot\,,B),B]_t = \mathrm{P}-\lim_{n\to\infty}\sum_{t_i\in D_t^n}\Big(F_x(t_i,B_{t_i})-F_x(t_{i-1},B_{t_{i-1}})\Big)\big(B_{t_i}-B_{t_{i-1}}\big)$$

and the set D_t^n consists of arbitrary points $0=t_0<t_1<\ldots<t_{n-1}<t_n=t$ satisfying $\max_{1\le i\le n}(t_i-t_{i-1})\to 0$ as $n\to\infty$.

5. **Eisenbaum's extension** [3]. Let $B=(B_t)_{t\ge 0}$ be a standard Brownian motion and let $F:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ be a measurable function satisfying the following conditions: (i) $t\mapsto F(t,x)$ is AC ; (ii) $x\mapsto F(t,x)$ is AC; (iii) $\int_0^t\int_{\mathbb{R}}|F_t(s,x)|\,(1/\sqrt{s})\,ds\,dx<\infty$; (iv) $\int_0^t\int_{\mathbb{R}}|F_x(s,x)|^2\,(1/\sqrt{s})\,\varphi(x/\sqrt{s})\,ds\,dx<\infty$; (v) $\int_0^t\int_{\mathbb{R}}|F_x(s,x)|\,(|x|/\sqrt{s})\,\varphi(x/\sqrt{s})\,ds\,dx<\infty$. [We recall that $\varphi(x)=(1/\sqrt{2\pi})\,\exp(-x^2/2)$ for $x\in\mathbb{R}$.] Then we have:

$$(2.8)\qquad F(t,B_t)=F(0,B_0)+\int_0^t F_t(s,B_s)\,ds+\int_0^t F_x(s,B_s)\,dB_s -\frac{1}{2}\int_0^t\int_{\mathbb{R}}F_x(s,x)\,d\ell_s^x$$

where ℓ_s^x is the local time of B at the point x given by (2.4) above, and $d\ell_s^x$ refers to an area integration with respect to $(s,x)\mapsto\ell_s^x$.

6. **Cherny's extension** [2]. Let $B=(B_t)_{t\ge 0}$ be a standard Brownian motion and let $F:\mathbb{R}\to\mathbb{R}$ be an AC function with F' being AC on $\mathbb{R}\setminus\{0\}$. Then:

$$(2.9)\qquad F(B_t)=F(B_0)+\int_0^t F'(B_s)\,dB_s+\frac{1}{2}\lim_{\varepsilon\downarrow 0}\,(F'(\varepsilon)-F'(-\varepsilon))\,\ell_t^0 +\frac{1}{2}\Big(v.p.\int_0^t F''(B_s)\,ds\Big)$$

whenever the principal value integral:

$$(2.10)\qquad v.p.\int_0^t F''(B_s)\,ds=\lim_{\varepsilon\downarrow 0}\int_0^t F''(B_s)\,I(|B_s|>\varepsilon)\,ds$$

exists as a limit in probability.

7. **Extension with local time on curves** [11]. Let $X=(X_t)_{t\ge 0}$ be a continuous semimartingale and let $b:\mathbb{R}_+\to\mathbb{R}$ be a continuous function of bounded variation. Setting $C=\{\,(s,x)\in\mathbb{R}_+\times\mathbb{R}\mid x<b(s)\,\}$ and $D=\{\,(s,x)\in\mathbb{R}_+\times\mathbb{R}\mid x>b(s)\,\}$ suppose that a continuous function $F:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is given such that F is $C^{1,2}$ on $\overline{C}$ and F is $C^{1,2}$ on $\overline{D}$. Then we have:

$$(2.11)\qquad F(t,X_t)=F(0,X_0)+\int_0^t F_t(s,X_s-)\,ds+\int_0^t F_x(s,X_s-)\,dX_s +\frac{1}{2}\int_0^t F_{xx}(s,X_s)\,I(X_s\ne b(s))\,d\langle X,X\rangle_s$$

$$+\frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\, I(X_s=b(s))\, d\ell_s^b$$

where ℓ_s^b is the local time of X at the curve b defined by:

$$\ell_s^b = I\!P-\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^s I(b(r)\le X_r< b(r)+\varepsilon)\, d\langle X,X\rangle_r \tag{2.12}$$

and $d\ell_s^b$ refers to the integration with respect to the continuous increasing function $s\mapsto \ell_s^b$. The formula (2.11) remains valid if the left-limits $F_t(s,X_s-)$ and $F_x(s,X_s-)$ are replaced by the right-limits $F_t(s,X_s+)$ and $F_x(s,X_s+)$ provided that $I(b(r)\le X_r<b(r)+\varepsilon)$ in (2.12) is replaced by $I(b(r)-\varepsilon\le X_r<b(r))$.

8. **Formal $d\ell_t^x$ extension** [Sections 3–5]. Let $X=(X_t)_{t\ge 0}$ be a continuous semimartingale and let $F: I\!R_+\times I\!R\to I\!R$ be a measurable function satisfying certain regularity conditions. Then we have:

$$F(t,X_t) = F(0,X_0) + \int_0^t D_tF(s,X_s)\, ds + \int_0^t D_xF(s,X_s)\, dX_s - \frac{1}{2}\int_0^t\int_{I\!R} D_xF(s,x)\, d\ell_s^x \tag{2.13}$$

where ℓ_s^x is the local time of X at the point x given by (2.4) above, and $d\ell_s^x$ refers to an area integration with respect to $(s,x)\mapsto \ell_s^x$. The operators D_t and D_x refer to a differentiation with respect to t and x , respectively.

9. **The occupation times formula** (cf. [12] or [6]). Let $X=(X_t)_{t\ge 0}$ be a continuous semimartingale. Comparing the final terms in (2.1) and (2.3) when F is C^2 and using the monotone class theorem (see e.g. [12] page 3) it follows:

$$\int_0^t G(X_s)\, d\langle X,X\rangle_s = \int_{I\!R} G(x)\, \ell_t^x\, dx \tag{2.14}$$

for every bounded measurable function $G: I\!R\to I\!R$. Similarly, inspecting first the case of $G=1_{(t_1,t_2]\times(x_1,x_2]}$ and then using the monotone class theorem, one gets the following extension of (2.14) to the time-dependent case:

$$\int_0^t G(s,X_s)\, d\langle X,X\rangle_s = \int_{I\!R}\Big(\int_0^t G(s,x)\, d_s\ell_s^x\Big) dx \tag{2.15}$$

for every bounded measurable function $G: I\!R_+\times I\!R\to I\!R$.

3. Local time-space formula

This section consists of two parts. In the first part we will define a local time-space integral of simple functions. In the second part we will extend it to more general functions.

1. Let $X=(X_t)_{t\ge 0}$ be a continuous semimaringale, and let ℓ_t^x be the local time of X at the point x given by (2.4) above.

With $t > 0$ given and fixed, let us introduce the following families of functions:

(3.1) $$\mathcal{P}_1 = \{1_{(t_1,t_2]}\, 1_{(x_1,x_2]} \mid 0 \le t_1 < t_2 \le t, -\infty < x_1 < x_2 < \infty\}$$

(3.2) $$\mathcal{P}_2 = \{C\, 1_{(x_1,x_2]} \mid C \in C^1([0,t]), -\infty < x_1 < x_2 < \infty\,\}$$

(3.3) $$\mathcal{P}_3 = \{C\, d \mid C \in C^1([0,t]), d \in C(I\!R)\,\}$$

(3.4) $$\mathcal{H}_1 = \{H \mid H = F_x \ \text{ for some } \ F \in C^1([0,t]\times I\!R)\,\}$$

(3.5) $$\mathcal{H}_2 = \{H \mid H = F_x \ \text{ for some } \ F \in C^2([0,t]\times I\!R)\,\}$$

(3.6) $$\mathcal{H}_{1,2} = \{H \mid H = F_x \ \text{ for some } \ F \in C^{1,2}([0,t]\times I\!R)\,\}$$

(3.7) $$\mathcal{M} = \{H \mid H : [0,t]\times I\!R \to I\!R \ \text{ is measurable}\,\}.$$

For $\mathcal{C} \subset \mathcal{M}$ let $sp(\mathcal{C})$ denote the smallest subspace of $\mathcal{M}$ containing $\mathcal{C}$. For example, we have:

(3.8) $$sp(\mathcal{P}_1) = \{\, \textstyle\sum_{i=1}^n \alpha_i\, 1_{(t_{i-1},t_i]}\, 1_{(x_{i-1},x_i]} \mid \alpha_1, \ldots, \alpha_n \in I\!R\,,\ n \ge 1\,\}$$

where there is no restriction to assume that $0 \le t_1 < t_2 < \ldots < t_n$ and $x_1 < x_2 < \ldots < x_n$.

Let L^0 denote the space of all random variables (defined on $(\Omega, \mathcal{F}, I\!P)$ where X is defined) equipped with the metric of convergence in $I\!P$-probability.

Define an operator $\Lambda : \mathcal{P}_1 \to L^0$ by setting:

(3.9) $$\Lambda(1_{(t_1,t_2]}\, 1_{(x_1,x_2]}) = \ell_{t_2}^{x_2} - \ell_{t_1}^{x_2} + \ell_{t_1}^{x_1} - \ell_{t_2}^{x_1}$$

and extend it by linearity to $sp(\mathcal{P}_1)$ as follows:

(3.10) $$\Lambda(\textstyle\sum_{i=1}^n \alpha_i\, 1_{(t_{i-1},t_i]}\, 1_{(x_{i-1},x_i]}) = \sum_{i=1}^n \alpha_i\, (\ell_{t_2}^{x_2} - \ell_{t_1}^{x_2} + \ell_{t_1}^{x_1} - \ell_{t_2}^{x_1})$$

both in reminiscence to the classic area integral in plane.

Let Λ also denote the linear extension of Λ from $sp(\mathcal{P}_1)$ to $\mathcal{D}(\Lambda)$ satisfying the following continuity condition:

(3.11) $$H_n \xrightarrow{\ell} H_0 \Rightarrow \Lambda(H_n) \to \Lambda(H_0) \ \text{ in } \ I\!P-\text{probability}$$

where $H_n \xrightarrow{\ell} H_0$ if and only if the following two conditions hold:

(3.12) $$H_n \to H_0 \ \text{ with } \ |H_n| \le G \ \text{ for all } \ n \ge 1 \ \text{ where } \ G \ \text{ is locally bounded}$$

(3.13) $$\partial_s \textstyle\int H_n\, dy \xrightarrow{\sim} \partial_s \int H_0\, dy$$

the meaning of (3.13) being that:

(3.14) $$\textstyle\int_0^t (\partial_s \int_0^x H_n(s,y)\, dy)\big|_{x=h(s)} \to \int_0^t (\partial_s \int_0^x H_0(s,y)\, dy)\big|_{x=h(s)}$$

for every continuous function $h : [0,t] \to I\!R$ as $n \to \infty$, where $\partial_s \int_0^x H_n(s,y)\, dy$ denotes the signed measure associated with the BV mapping $s \mapsto \int_0^x H_n(s,y)\, dy$ for $n \ge 0$.

The question then arises to determine if there exists such an extension of Λ and if it is unique. More precisely, it is of interest to determine (characterize) the maximal subspace of $\mathcal{M}$ to which Λ can be extended uniquely. This subspace is then denoted by $\mathcal{D}(\Lambda)$ and is called the (maximal) domain of Λ .

We will not deal with the latter question in full generality. Instead we will display subspaces of $\mathcal{M}$ to which Λ can be extended uniquely and its action determined explicitly. In this section we will show that $\mathcal{H}_1$ from (3.4) above is one such subspace (with a further subspace $\mathcal{H}_{1,2}$ of particular interest). In the next section we will make attempts to go beyond the subspace $\mathcal{H}_1$ at some moderate depth.

2. It is naturally guessed using (3.9) that $\mathcal{P}_2 \subset \mathcal{D}(\Lambda)$ and that we have:

$$\Lambda(C\,1_{(x_1,x_2]}) = \int_0^t C(s)\,(d_s\ell_s^{x_2} - d_s\ell_s^{x_1}) \tag{3.15}$$

for $C\,1_{(x_1,x_2]}$ from $\mathcal{P}_2$. A formal verification can be carried out using that $sp(\mathcal{P}_1)$ is dense in $\mathcal{P}_2$ relative to the ℓ-convergence in (3.11) (recalling also that $s \mapsto \ell_s^{x_i}$ is increasing and continuous). We take (3.11) to be the initial point in the proof of the following theorem.

Theorem 3.1. *There exists a unique linear extension of Λ from $sp(\mathcal{P}_1)$ to $sp(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{H}_1)$ satisfying (3.10) and (3.11). In view of (3.9) this extension will be denoted by either of the area integrals:*

$$\Lambda(H) = \int_0^t \int_{I\!R} H(s,x)\,d\ell_s^x = \int_{I\!R} \int_0^t H(s,x)\,d\ell_s^x \tag{3.16}$$

where the order of integration is interchanged formally.

Proof. The key argument in the proof will be provided using the following extension of the formula (2.11) to the case when instead of one function b we are given finitely many functions $b_1,\dots,b_n$ which do not intersect (cf. Remarks 3.2 and 3.3 in [11]).

More precisely, let us assume that the following conditions are satisfied:

$$b_i : I\!R_+ \to I\!R \text{ is continuous and of bounded variation for } 1 \le i \le n \tag{3.17}$$

$$F_i : I\!R_+ \times I\!R \to I\!R \text{ is } C^{1,2} \text{ for } 1 \le i \le n+1 \tag{3.18}$$

$$\begin{aligned} F(t,x) &= F_1(t,x) \text{ if } x < b_1(t) \\ &= F_i(t,x) \text{ if } b_{i-1}(t) < x < b_i(t) \text{ for } 2 \le i \le n \\ &= F_{n+1}(t,x) \text{ if } x > b_n(t) \end{aligned} \tag{3.19}$$

where $F : I\!R_+ \times I\!R \to I\!R$ is continuous. Then (2.11) extends as follows:

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t F_t(s,X_s-)\,ds + \int_0^t F_x(s,X_s-)\,dX_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\,I\big(X_s \notin \{b_1(s),\dots,b_n(s)\}\big)\,d\langle X,X\rangle_s \\ &+ \frac{1}{2}\sum_{i=1}^n \int_0^t \Big(F_x(s,X_s+) - F_x(s,X_s-)\Big)\,I(X_s = b_i(s))\,d\ell_s^{b_i}(X) \end{aligned} \tag{3.20}$$

where $\ell_s^{b_i}$ is the local time of X at the curve b_i given by (2.12) above, and $d\ell_s^{b_i}(X)$ refers to the integration with respect to $s \mapsto \ell_s^{b_i}(X)$ for $i = 1,\dots,n$. [The formula (3.20) remains valid if the left-limits $F_t(s,X_s-)$ and $F_x(s,X_s-)$ are replaced by the right-limits $F_t(s, X_s+)$ and $F_x(s, X_s+)$ provided that $I(b_i(r) \le X_r < b_i(r)+\varepsilon)$ in the definition (2.12) is replaced by $I(b_i(r)-\varepsilon \le X_r < b_i(r))$ for $i = 1, \dots, n$.]

1. Take $C\, 1_{(x_1,x_2]}$ from $\mathcal{P}_2$, set $d(x) = 1_{(x_1,x_2]}(x)$, and let $D(x) = \int_0^x d(y)\, dy$. Setting $b_1(s) \equiv x_1$, $b_2(s) \equiv x_2$, $F_1(s,x) \equiv 0$, $F_2(s,x) = C(s)(x - x_1)$ and $F_3(s,x) = C(s)(x_2 - x_1)$ we see that (3.17)-(3.19) are satisfied so that (3.20) gives:

$$
\begin{aligned}
(3.21) \qquad C(t)D(X_t) = C(0)D(X_0) + \int_0^t C'(s)D(X_s)\, ds + \int_0^t C(s)d(X_s)\, dX_s \\
+ \frac{1}{2}\Big(\int_0^t C(s)\, d_s\ell_s^{x_1} - \int_0^t C(s)\, d_s\ell_s^{x_2} \Big)
\end{aligned}
$$

using that $F_x(s, b_1(s)+) - F_x(s, b_1(s)+) = C(s)\,(d(x_1+) - d(x_1-)) = C(s)$ and $F_x(s, b_2(s)+) - F_x(s, b_2(s)+) = C(s)\,(d(x_2+) - d(x_2-)) = -C(s)$.

In view of (3.15) above, the final term in (3.21) may be recognized as a negative action of the operator Λ on $C\, 1_{(x_1,x_2]}$, or in other words:

$$
\begin{aligned}
(3.22) \qquad C(t)D(X_t) = C(0)D(X_0) + \int_0^t C'(s)D(X_s)\, ds \\
+ \int_0^t C(s)\, d(X_s)\, dX_s - \frac{1}{2}\, \Lambda(C\, d)
\end{aligned}
$$

yielding a stochastic-integral representation for the action of Λ on $\mathcal{P}_2$.

2. Given $c = 1_{(t_1,t_2]}$ with $0 \le t_1 < t_2 \le t$ introduce the convolution approximation by defining a sequence of functions C_n with $n \ge 1$ as follows:

$$
(3.23) \qquad C_n(s) = \int_{I\!R_+} c(s - r/n)\, \Omega(r)\, dr
$$

where $\Omega : I\!R \to I\!R$ is C^∞ with $\mathrm{supp}(\Omega) = [0,1]$ and $\int_{I\!R} \Omega(r)\, dr = 1$. Then $C_n : [0,t] \to I\!R$ is C^∞ and $C_n \to c$ as $n \to \infty$ with $|C_n| \le 1$ for all $n \ge 1$. Moreover $C_n' \overset{\sim}{\to} \delta_{t_1} - \delta_{t_2}$ i.e. $\int_0^t h(s)\, dC_n(s) \to h(t_1) - h(t_2)$ for every continuous function $h : [0,t] \to I\!R$ as $n \to \infty$.

Inserting C_n in place of C in (3.22) and letting $n \to \infty$, using the convergence relations just exhibited and the stochastic dominated convergence theorem (see e.g. [12] page 142), as well as (3.11) above, we get:

$$
\begin{aligned}
(3.24) \qquad c(t)D(X_t) = c(0)D(X_0) + (D(X_{t_1}) - D(X_{t_2})) \\
+ \int_0^t c(s)d(X_s)\, dX_s - \frac{1}{2}\, \Lambda(c\, d)
\end{aligned}
$$

yielding a stochastic-integral representation for the action of Λ on $\mathcal{P}_1$.

3. Take $C\, d$ from $\mathcal{P}_3$ and choose a sequence of simple functions:

$$
(3.25) \qquad d_n = \sum_{i=1}^{k_n} \beta_i^n\, 1_{(x_{i-1}^n, x_i^n]}
$$

so that $d_n \to d$ as $n \to \infty$ with $|d_n| \le g$ for all $n \ge 1$ where g is locally bounded. Setting $D_n(x) = \int_0^x d_n(y)\,dy$ we also have $D_n \to D$ as $n \to \infty$ with $|D_n| \le \Gamma$ for all $n \ge 1$ where Γ is locally bounded.

Inserting D_n in place of D in (3.22) above, and likewise d_n in place of d , letting $n \to \infty$ and using the dominated convergence theorem (both deterministic and stochastic) and localization, as well as (3.11) above, it follows that (3.22) extends to all $C \in C^1(\mathbb{R}_+)$ and $D \in C^1(\mathbb{R})$. In particular, it shows that (3.22) holds for all polynomials in two variables s and x .

4. Take H from $\mathcal{H}_2$ and let F be from $C^2([0,t]\times\mathbb{R})$ such that $H = F_x$. For this F there exists a sequence of polynomials $P^n(s,x) = \sum_{i=1}^{k_n} C^{i,n}(s)D^{i,n}(x)$ such that $P^n \to F$, $P_t^n \to F_t$ and $P_x^n \to F_x$ uniformly on a compact set in $\mathbb{R}_+\times\mathbb{R}$ as $n \to \infty$ [for this use that F_{tx} is continuous and find a sequence of polynomials $\widetilde{P}^n$ such that $\widetilde{P}^n \to F_{tx}$ on a given compact set by means of the Weierstrass theorem; the sequence $P^n(s,x) = \int_0^s \int_0^x \widetilde{P}^n\,drdy$ then has the desired properties]. Extending then (3.22) by linearity from $C(s)D(x)$ to $P^n(s,x)$, letting $n \to \infty$ in the resulting formula, and using the dominated convergence theorem (both deterministic and stochastic) and localization, as well as (3.11) above, we obtain:

$$\begin{aligned}(3.26)\qquad \int_0^{X_t} H(t,y)\,dy = \int_0^{X_0} H(0,y)\,dy + \int_0^t \Big(\frac{\partial}{\partial s}\int_0^x H(s,y)\,dy\Big)\Big|_{x=X_s} ds \\ + \int_0^t H(s,X_s)\,dX_s - \frac{1}{2}\,\Lambda(H)\end{aligned}$$

yielding a stochastic-integral representation for the action of Λ on $\mathcal{H}_2$.

5. Take H from $\mathcal{H}_1$ and let F be from $C^1([0,t]\times\mathbb{R})$ such that $H = F_x$. Associate with this F the convolution approximation:

$$(3.27)\qquad F^n(s,x) = \int_{\mathbb{R}_+}\int_{\mathbb{R}} F(s-r/n, x-y/n)\,\Omega(r)\,\Omega(y)\,dr\,dy$$

where Ω is the same as in (3.23) above. Then F^n is C^∞ and we have $F^n \to F$, $F_t^n \to F_t$, $F_x^n \to F_x$ uniformly on a compact set in $\mathbb{R}_+\times\mathbb{R}$ as $n \to \infty$.

Setting $H^n = F_x^n$ and inserting this H^n in place of H in (3.26) above, letting $n \to \infty$ and using the dominated convergence theorem (both deterministic and stochastic) and localization, as well as (3.11) above, it follows that (3.26) extends to all $H \in \mathcal{H}_1$.

6. Combining this fact with (3.22) and (3.24) we see that the stochastic-integral representation (3.26) (with $(\partial/\partial s)ds$ replaced by ∂_s when needed) for the action of Λ on H holds for all $H \in sp(\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_3\cup\mathcal{H}_1)$ from where by (3.12)-(3.14) using the dominated convergence theorem (both deterministic and stochastic) it follows that (3.11) holds on the entire $sp(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{H}_1)$. This shows that the definition of Λ is not dependent on the particular choice of the approximating sequence used above, and the proof of the theorem is complete.

$\square$

Corollary 3.2 (**Local time-space formula**). *Let $X = (X_t)_{t \ge 0}$ be a continuous semimartingale and let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a C^1 function. Then the following change-of-variable formula holds:*

$$
\begin{aligned}
F(t, X_t) = & F(0, X_0) + \int_0^t F_t(s, X_s)\, ds + \int_0^t F_x(s, X_s)\, dX_s \\
& - \frac{1}{2} \int_0^t \int_{\mathbb{R}} F_x(s, x)\, d\ell_s^x
\end{aligned}
\tag{3.28}
$$

where ℓ_s^x is the local time of X at the point x given by (2.4) above, and $d\ell_s^x$ refers to the area integration with respect to $(s, x) \mapsto \ell_s^x$ established in Theorem 3.1.

Proof. It follows by (3.16) and (3.26) above.

□

Corollary 3.3 (**Integration by parts**). *If $H \in \mathcal{H}_{1,2}$ then we have:*

$$
\int_{\mathbb{R}} \int_0^t H(s, x)\, d\ell_s^x = - \int_{\mathbb{R}} \left(\int_0^t H_x(s, x)\, d_s \ell_s^x \right) dx
\tag{3.29}
$$

and if $H \in \mathcal{H}_2$ then we have:

$$
\int_0^t \int_{\mathbb{R}} H(s, x)\, d\ell_s^x = \int_{\mathbb{R}} H(t, x)\, d_x \ell_t^x - \int_0^t \left(\int_{\mathbb{R}} H_t(s, x)\, d_x \ell_s^x \right) ds
\tag{3.30}
$$

where $d\ell_s^x$ refers to the area integration with respect to $(s, x) \mapsto \ell_s^x$ established in Theorem 3.1.

Proof. The first identity follows by comparing (3.28) with (2.1) and using (2.15). [It also follows by a formal partial integration upon setting $u(x) = H(s, x)$ and $dv(x) = d_x(\int_0^t d_s \ell_s^x)$ so that $du(x) = H_x(s, x)\, dx$ and $v(x) = \int_0^t d_s \ell_s^x$ with $\ell_s^{\pm\infty} \equiv 0$.] The second identity follows by a formal partial integration upon setting $u(s) = H(s, x)$ and $dv(s) = d_s(\int_{\mathbb{R}} d_x \ell_s^x)$ so that $du(s) = H_t(s, x)\, ds$ and $v(s) = \int_{\mathbb{R}} d_x \ell_s^x$ with $\int_{\mathbb{R}} d_x \ell_0^x \equiv 0$. [It can be justified using (2.5) above but we will omit the details.]

□

In contrast to the semimartingale decomposition of $F(t, X_t)$ in Itô's formula (2.1) when F is $C^{1,2}$, it is important to realize that the final term in (3.28) is not necessarily of bounded variation in t when F is C^1 only. [In dimension one the same remark holds for the formulas (2.3) and (2.5), respectively.]

A comparison of (3.28) with (2.6)+(2.7) suggests that Λ admits a representation as the difference of a forward and backward stochastic integral at least when X is time-reversal invariant such as a Brownian motion process (cf. [3]). We will not examine this interesting point any further in the present article.

4. Extended local time-space formula

The level of generality reached in the preceding section by extending the operator Λ to $\mathcal{H}_1$ is somewhere between (2.1) and (2.3). It is thus natural to pursue further extensions so to recover the Itô-Tanaka formula (2.3) as a special case in dimension one.

Convexity in two dimensions is more complicated and there seem to be no simple $AC+BV$ characterization as in dimension one according to which $F = F_1 - F_2$ where F_1 and F_2 are convex if and only if F is AC with F' of BV . Moreover, considering the setting of (2.11) above and letting b to oscillate wildly (being also a more general curve in $I\!R_+ \times I\!R$ rather than just a function of time) we see that the assumption of the existence of the one-sided limits $F_t(s-, x-)$ and $F_x(s-, x-)$ may generally be rather restrictive. It is however exceedingly difficult to incorporate all special cases under a general condition which is technically not too demanding. We therefore state the theorem below more as an indication of what can be done in special cases than to give a final word on all possible extensions.

In the setting of Section 3 above, with $t > 0$ given and fixed, let us denote by $C^1_-([0,t] \times I\!R)$ the family of functions $F : [0,t] \times I\!R \to I\!R$ satisfying the following conditions:

$$\text{(4.1)} \qquad \text{limits } F_t(s-, x-) \text{ and } F_x(s-, x-) \text{ exist at all } (s,x) \in [0,t] \times I\!R$$

$$\text{(4.2)} \qquad (s,t) \mapsto F_t(s-, x-) \text{ and } (s,t) \mapsto F_x(s-, x-) \text{ are locally bounded on } [0,t] \times I\!R.$$

Extend the family $\mathcal{H}_1$ from (3.4) as follows:

$$\text{(4.3)} \qquad \mathcal{H}_1^- = \{\, H \mid H(s,x) = F_x(s-, x-) \text{ for some } F \in C^1_-([0,t] \times I\!R) \,\}.$$

Denoting $\mathcal{C} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{H}_1$ we then have the following sequel to Theorem 3.1 above.

Theorem 4.1. *There exists a unique linear extension of Λ from $sp(\mathcal{C})$ to $sp(\mathcal{C} \cup \mathcal{H}_1^-)$ satisfying (3.10) and (3.11). In view of (3.9) this extension will also be denoted by (3.16) above.*

Proof. Take $H \in \mathcal{H}_1^-$ and let F be from $C^1_-([0,t] \times I\!R)$ such that $H(s,x) = F_x(s-, x-)$ for all $0 \le s \le t$ and all $x \in I\!R$. Associate with this F the convolution approximation F^n from (3.27) above. Then F^n is C^∞ and $F^n(s,x) \to F(s,x)$, $F^n_t(s,x) \to F_t(s-, x-)$ and $F^n_x(s,x) \to F_x(s-, x-)$ for all $(s,x) \in [0,t] \times I\!R$. Moreover, by means of (4.2) we can achieve that $|F^n_t| \le \gamma$ and $|F^n_x| \le G$ for all $n \ge 1$ where γ and G are locally bounded on $[0,t] \times I\!R$.

Setting $H^n = F^n_x$ and inserting this H^n in place of H in (3.26) above, letting $n \to \infty$ and using the dominated convergence theorem (both deterministic and stochastic) and localization, as well as (3.11) above, it follows that (3.26) extends to all $H \in \mathcal{H}_1^-$.

$\square$

Corollary 4.2 (**Extended local time-space formula).** *Let $X = (X_t)_{t\ge 0}$ be a continuous semimartingale and let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function from the class $C^1_-([0,t]\times\mathbb{R})$ with $t>0$. Then the following change-of-variable formula holds:*

$$F(t,X_t) = F(0,X_0) + \int_0^t F_t(s-,X_s-)\,ds + \int_0^t F_x(s-,X_s-)\,dX_s - \frac{1}{2}\int_0^t \int_{\mathbb{R}} F_x(s-,x-)\,d\ell_s^x \tag{4.4}$$

where ℓ_s^x is the local time of X at the point x given by (2.4) above, and $d\ell_s^x$ refers to the area integration with respect to $(s,x)\mapsto \ell_s^x$ established in Theorem 4.1.

Proof. It follows from the proof of Theorem 4.1 above.

□

Corollary 4.3 (**Integration by parts for products).** *If $C : \mathbb{R}_+ \to \mathbb{R}$ and $D : \mathbb{R}\to\mathbb{R}$ are of BV then we have:*

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t C(s)D(x)\,d\ell_s^x &= -\int_{\mathbb{R}}\Big(\int_0^t C(s)\,d_s\ell_s^x\Big)\,dD(x) \\ &= C(t)\int_{\mathbb{R}} D(x)\,d_x\ell_t^x - \int_0^t\Big(\int_{\mathbb{R}} D(x)\,d_x\ell_s^x\Big)\,dC(s) \\ &= -C(t)\int_{\mathbb{R}} \ell_t^x\,dD(x) + \int_0^t\Big(\int_{\mathbb{R}} \ell_s^x\,dD(x)\Big)\,dC(s)\end{aligned} \tag{4.5}$$

where $d\ell_s^x$ refers to the area integration with respect to $(s,x)\mapsto\ell_s^x$ established in Theorem 4.1.

Proof. It follows by combining the results of Corollary 4.2 and Corollary 3.3.

□

In particular, when $C(s)\equiv 1$ and $F_x(s-,x-) = D(x)$ then the formula (4.4) above together with the final identity in (4.5) reduces to the Itô-Tanaka formula (2.3).

In a general case of (3.29) the first identity in (4.5) can be written as follows:

$$\int_{\mathbb{R}}\int_0^t H(s,x)\,d\ell_s^x = -\int_{\mathbb{R}}\int_0^t d_xH(s,x)\,d_s\ell_s^x \tag{4.6}$$

but it may be not so obvious to determine its sense.

Clearly, replacing the left-limits in (4.1)-(4.3) above with the right-limits it follows that Theorem 4.1, Corollary 4.2 and Corollary 4.3 extend to the case where $\mathcal{H}_1^-$ is replaced by $\mathcal{H}_1^+$ provided that $I(x\le X_s < x+\varepsilon)$ in the definition (2.4) is replaced by $I(x-\varepsilon < X_s \le x)$.

5. Formal $d\ell_t^x$ calculus

In this section we will show how extensions of Itô's formula reviewed in Section 2 above can be obtained by formal manipulations of the $d\ell_s^x$ integral in the formula (2.13) which we recall here:

$$F(t,X_t) = F(0,X_0) + \int_0^t D_t F(s,X_s)\,ds + \int_0^t D_x F(s,X_s)\,dX_s - \frac{1}{2}\int_0^t \int_{I\!R} D_x F(s,x)\,d\ell_s^x. \tag{5.1}$$

The formulas (3.28) and (4.4) derived above are special cases of this general formula.

This formalism appears to be useful for at least two reasons. Firstly, it helps to develop intuition needed to understand and compare known formulas. Secondly, if a new function F is given and one needs to write down a change-of-variable formula for $F(t,X_t)$, then such a formalism can be helpful in guessing a candidate formula before a rigorous proof is found. This will be illustrated in the final example below.

Let us now apply the formal $d\ell_t^x$ calculus and show how formulas (2.3), (2.5), (2.9) and (2.11) follow from (5.1). We assume throughout that $X = (X_t)_{t\ge0}$ is a continuous semimartingale and F is a given function defined on $I\!R_+$ or $I\!R_+ \times I\!R$ with values in $I\!R$. Relevant properties of F will be inherited from Section 2 without further mentioning.

1. *Formulas (2.3) and (2.5).* In this case (5.1) reads as follows:

$$F(X_t) = F(X_0) + \int_0^t F'_-(X_s)\,dX_s - \frac{1}{2}\int_{I\!R} F'_-(x)\,d_x\ell_t^x \tag{5.2}$$

which (at least formally) coincides with the formula (2.5). To obtain (2.3) let us perform a formal partial integration in the final integral upon setting $u(x) = F'_-(x)$ and $dv(x) = d_x\ell_t^x$ so that $du(x) = dF'_-(x)$ and $v(x) = \ell_t^x$. The final integral in (5.2) then transforms as follows:

$$\int_{I\!R} F'_-(x)\,d_x\ell_t^x = F'_-(x)\,\ell_t^x\,\Big|_{x=-\infty}^{\infty} - \int_{I\!R} \ell_t^x\,dF'_-(x) = -\int_{I\!R} \ell_t^x\,dF'(x) \tag{5.3}$$

where we use that $\ell_t^{\pm\infty} \equiv 0$. Inserting (5.2) into (5.1) we formally obtain (2.3).

2. *Formula (2.9).* In this case (5.1) reads as follows:

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\,dX_s - \frac{1}{2}\int_{I\!R} F'(x)\,d_x\ell_t^x. \tag{5.4}$$

The final integral in (5.4) can be written as:

$$\int_{I\!R} F'(x)\,d_x\ell_t^x = \int_{\{|x|\le\varepsilon\}} F'(x)\,d_x\ell_t^x + \int_{\{|x|>\varepsilon\}} F'(x)\,d_x\ell_t^x. \tag{5.5}$$

A formal partial integration in the first integral on the right-hand-side upon setting $u(x) = F'(x)$ and $dv(x) = d_x \ell_t^x$ so that $du(x) = dF'(x)$ and $v(x) = \ell_t^x$ yields:

$$\tag{5.6} \begin{aligned} \int_{\{|x|\le\varepsilon\}} F'(x)\, d_x\ell_t^x &= \Big(\int_{(-\varepsilon,0)} + \int_{\{0\}} + \int_{(0,\varepsilon)} \Big) F'(x)\, d_x \ell_t^x \\ &= F'(x)\, \ell_t^x \left(\Big|_{x=-\varepsilon}^{0-} + \Big|_{x=0}^{0} + \Big|_{x=0+}^{\varepsilon} \right) - \int_{(-\varepsilon,0)\cup(0,\varepsilon)} \ell_t^x \, dF'(x) \\ &- \int_{\{0\}} \ell_t^x \, dF'(x) \rightarrow (-\ell_t^0)\, (F'(0+) - F'(0-)) \end{aligned}$$

as $\varepsilon \downarrow 0$. Moreover, using (3.29) and (2.14) we see that the second integral transforms as follows:

$$\tag{5.7} \begin{aligned} \int_{\{|x|>\varepsilon\}} F'(x)\, d_x \ell_t^x &= - \int_{\{|x|>\varepsilon\}} F''(x)\, \ell_t^x \, dx \\ &= - \int_0^t F''(X_s)\, I(|X_s| > \varepsilon)\, d\big\langle X, X \big\rangle_s . \end{aligned}$$

Letting $\varepsilon \downarrow 0$ in (5.5) using (5.6) and (5.7), where $d\big\langle X, X\big\rangle_s = ds$ when X is standard Brownian motion, and inserting the resulting identity in (5.4), we formally obtain (2.9).

3. *Formula (2.11).* In this case (5.1) reads as follows:

$$\tag{5.8} \begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t F_t(s, X_s-)\, ds + \int_0^t F_x(s, X_s-)\, dX_s \\ &- \frac{1}{2} \int_0^t \int_{I\!R} F_x(s, x-)\, d\ell_s^x . \end{aligned}$$

The final integral in (5.9) can be written as:

$$\tag{5.9} \begin{aligned} \int_0^t \int_{I\!R} F_x(s, x-)\, d\ell_s^x &= \int_0^t \int_{I\!R \setminus \{b(s)\}} F_x(s, x-)\, d\ell_s^x \\ &+ \int_{\{b(s)\}} F_x(s, x-)\, d_x \Big(\int_0^t d_s \ell_s^x \Big). \end{aligned}$$

Using (3.29) and (2.15) in the first integral on the right-hand-side and a formal partial integration in the second integral upon setting $u(x) = F_x(s, x-)$ and $dv(x) = d_x(\int_0^t d_s \ell_s^x)$ so that $du(x) = d_x F_x(s, x-)$ and $v(x) = \int_0^t d_s \ell_s^x$, we get:

$$\tag{5.10} \begin{aligned} \int_0^t \int_{I\!R} F_x(s, x-)\, d\ell_s^x &= - \int_0^t \Big(\int_{I\!R\setminus\{b(s)\}} F_{xx}(s, x) d_s \ell_s^x \Big) dx \\ &+ \Big(F_x(s, x-) \int_0^t d_s \ell_s^x \Big) \Big|_{x=b(s)}^{b(s)} - \int_{\{b(s)\}} \Big(\int_0^t d_s \ell_s^x \Big)\, d_x F_x(s, x-) \\ &= - \int_0^t F_{xx}(s, X_s)\, I(X_s \ne b(s))\, d\big\langle X, X \big\rangle_s \end{aligned}$$

$$- \int_0^t \Big(F_x(s, b(s)+) - F_x(s, b(s)-) \Big)\, d_s \ell_s^b$$

upon a formal identification $\int_0^t d_s \ell_s^{b(s)} = \int_0^t d_s \ell_s^b$. Inserting the resulting identity (5.10) into (5.8) we formally obtain (2.11).

4. Finally, let us briefly examine a new case not covered by known formulas from Section 2. For this, let us consider the setting of (2.11) where instead of a function $b : I\!R_+ \to I\!R$ of time we are given a function $c : I\!R \to I\!R$ of space. Setting $C = \{ (s,x) \in I\!R_+ \times I\!R \mid s < c(x) \}$ and $D = \{ (s,x) \in I\!R_+ \times I\!R \mid s > c(x) \}$ suppose that a continuous function $F : I\!R_+ \times I\!R \to I\!R$ is given such that F is $C^{1,2}$ on $\overline{C}$ and F is $C^{1,2}$ on $\overline{D}$. The question then arises to write down a change-of-variable formula for $F(t, X_t)$.

In this case (5.1) reads as follows:

$$F(t,X_t) = F(0,X_0) + \int_0^t F_t(s-, X_s)\, ds + \int_0^t F_x(s-, X_s)\, dX_s - \frac{1}{2} \int_{I\!R} \int_0^t F_x(s-, x)\, d\ell_s^x. \tag{5.11}$$

The final integral in (5.11) can be written as:

$$\int_{I\!R} \int_0^t F_x(s-, x)\, d\ell_s^x = \int_{I\!R} \int_{[0,t]\setminus\{c(x)\}} F_x(s-, x)\, d\ell_s^x + \int_{\{c(x)\}} F_x(s-, x)\, d_s \Big(\int_{I\!R} d_x \ell_s^x \Big). \tag{5.12}$$

Using (3.29) and (2.15) in the first integral on the right-hand-side and a formal partial integration in the second integral upon setting $u(s) = F_x(s-, x)$ and $dv(s) = d_s(\int_{I\!R} d_x \ell_s^x)$ so that $du(s) = d_s F_x(s-, x)$ and $v(s) = \int_{I\!R} d_x \ell_s^x$, we get:

$$\begin{aligned} \int_{I\!R} \int_0^t F_x(s-, x)\, d\ell_s^x &= - \int_{I\!R} \Big(\int_{[0,t]\setminus\{c(x)\}} F_{xx}(s,x) d_s d\ell_s^x \Big) dx \\ &\quad + \Big(F_x(s-, x) \int_{I\!R} d_x \ell_s^x \Big) \Big|_{x=c(x)}^{c(x)} - \int_{\{c(x)\}} \Big(\int_{I\!R} d_x \ell_s^x \Big)\, d_s F_x(s-, x) \\ &= - \int_0^t F_{xx}(s, X_s)\, I(s \neq c(X_s))\, d\langle X, X \rangle_s \\ &\quad - \int_{I\!R} \Big(F_x(c(x)+, x) - F_x(c(x)-, x) \Big)\, d_x \ell_c^x \end{aligned} \tag{5.13}$$

upon a formal identification $\int_0^t d_x \ell_{c(x)}^x = \int_0^t d_x \ell_c^x$. To give sense to the latter integral introduce:

$$\ell_c^x = I\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{c^{-1}([0,t])} I(x \le X_{c(x)} < x + \varepsilon)\, d\langle X, X \rangle_{c(x)}. \tag{5.14}$$

Then the final integral in (5.13) may be interpreted as the integral with respect to $x \mapsto \ell_c^x$, and inserting (5.13) into (5.11), we obtain the following candidate for the change-of-variable formula:

$$
\begin{aligned}
(5.15) \qquad F(t,X_t) =&F(0,X_0) + \int_0^t F_t(s-,X_s)\,ds + \int_0^t F_x(s-,X_s)\,dX_s \\
&+ \frac{1}{2}\int_0^t F_{xx}(s,X_s) I(s \neq c(X_s)) d\langle X,X\rangle_s \\
&+ \frac{1}{2}\int_{I\!R} \Big(F_x(c(x)+,x) - F_x(c(x)-,x) \Big)\, d_x \ell_c^x
\end{aligned}
$$

In order to view the final integral differently note that:

$$
(5.16) \qquad \ell_c^x = I\!P\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^t I(c^{-1}(s) \le X_s < c^{-1}(s)+\varepsilon)\, d\langle X,X\rangle_s
$$

where $c^{-1}(s) = \{\, x \in I\!R \mid c(x) = s \,\}$ and $I(c^{-1}(s) \le X_s < c^{-1}(s)+\varepsilon)$ denotes the corresponding sum of $I(x \le X_s < x+\varepsilon)$ over x running through $c^{-1}(s)$.

Similar candidate formulas can be obtained when b or c above is a curve in $I\!R_+ \times I\!R$ defined through a mapping $\gamma : [0,1] \to I\!R_+ \times I\!R$, but we will omit the details. It is an interesting problem to establish these formulas rigorously under natural conditions.

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