Affine random equations and the stable $(\frac{1}{2})$ distribution

C. Donati-Martin, R. Ghomrasni and M.Yor*

1 Introduction and main result

In various probabilistic areas of research, affine random equations are studied, i.e. given a pair (A, B) of random variables, one is interested in the study of all possible variables X such that :

$$X \stackrel{(law)}{=} A + BX, \tag{1.1}$$

where, on the right-hand side, X is independent from the pair (A, B). (See, e.g., Babillot, Bougerol and Elie [?] for some recent study in the so-called critical case, and the references therein).
Converse studies, for which the law of X is given a priori, and one looks for all possible pairs (A, B) of random variables satisfying (??) seem to be less popular. In the present note, we discuss the important particular case of such a converse study when $X \equiv T$ is the stable $(\frac{1}{2})$ variable, i.e. :

$$P(T \in dt) = \frac{dt}{\sqrt{2\pi t^3}} \exp(-\frac{1}{2t}).$$

More precisely, we are interested in giving a description of all possible pairs (S, L) of random variables taking values in $\mathbb{R}_+$ such that :

$$T \stackrel{(law)}{=} S + L^2 T. \tag{1.2}$$

*Laboratoire de Probabilités et Modèles aléatoires - UMR CNRS 7599, Tour 56, 3^e étage, Université Paris VI, 4, place Jussieu - 75252 Paris Cedex 05

Our motivation to study this particular converse equation is that it arises very naturally when dealing with strictly positive continuous local martingales $(M_t, t \geq 0)$ converging to 0 as $t \to \infty$. More precisely, one has the following

Lemma 1.1 *Let $(M_t; t \geq 0)$ be a $\mathbb{R}_+$-valued, continuous local $(\mathcal{F}_t)$ martingale, such that $M_0 = 1$, and $M_t \to_{t\to\infty} 0$, and let θ be any finite $(\mathcal{F}_t)$ stopping time. Then, the pair $(S = \langle M \rangle_\theta, L = M_\theta)$ solves (**??**).*

Proof: First, we remark that the Dubins-Schwarz representation of (M_t) : $M_t = \beta_{\langle M \rangle_t}$, $t \geq 0$, where $(\beta_u, u \geq 0)$ is a Brownian motion, implies that $\langle M \rangle_\infty \overset{(law)}{=} T$. Applying the same argument to $(M_{\theta+u}, u \geq 0)$, conditionally on $\mathcal{F}_\theta$, and writing :

$$\langle M \rangle_\infty = \langle M \rangle_\theta + (\langle M \rangle_\infty - \langle M \rangle_\theta),$$

we obtain that :

$$T \overset{(law)}{=} \langle M \rangle_\theta + (M_\theta)^2 T,$$

where, on the right-hand side, T is independent of the pair $(\langle M \rangle_\theta, M_\theta)$. $\square$

Remark: The identity (**??**) may also be presented in a more analytic manner as follows :

$$\text{for every } \lambda \geq 0 , \qquad E[\exp(-(\frac{\lambda^2}{2} S + \lambda L))] = \exp(-\lambda). \tag{1.3}$$

Of course, this agrees with the well known fact that, under the hypothesis of Lemma 1,

$$\exp(-\lambda M_{t\wedge\theta} - \frac{\lambda^2}{2} \langle M \rangle_{t\wedge\theta})$$

is a bounded martingale; hence:

$$E[\exp(-\lambda M_\theta - \frac{\lambda^2}{2} \langle M \rangle_\theta)] = \exp(-\lambda). \tag{1.4}$$

The main result of this Note is the following:

Theorem 1.1 *Let $L \geq 0$. In order that, on an adequate probability space, a variable S may be constructed such that (S, L) satisfies (**??**), it is necessary and sufficient that :*

$$E(L) \leq 1. \tag{1.5}$$

Moreover, if $E(S) < \infty$, then $E(L) = 1$.

Proof: a) First, we assume that (S, L) satisfies (**??**). Then, from (**??**), we deduce : for any $\lambda \geq 0$,

$$\exp(-\lambda) \leq E(\exp(-\lambda L)],$$

or equivalently:

$$1 - E(\exp(-\lambda L)] \leq 1 - \exp(-\lambda)$$

Then, we write:

$$1 - \exp(-x) = x \int_0^1 dy \exp(-xy),$$

and we deduce:

$$E[L \int_0^1 dy \exp(-\lambda L y)] \leq \frac{1 - \exp(-\lambda)}{\lambda}.$$

Letting λ decrease to 0, we obtain as a consequence of the Beppo-Levi theorem: $E(L) \leq 1$.

b) Conversely, let us assume: $E(L) = c \leq 1$. Denote by μ the law of $(L - c)$; μ is carried by $[-c, \infty)$, and it satisfies: $\int x\mu(dx) = 0$.
Next, we consider any procedure leading to a solution of Skorokhod's embedding problem relative to μ, that is precisely any stopping time T_μ in the filtration of $(B_t; t \geq 0)$ a real valued Brownian motion such that :

i) the law of B_{T_μ} is μ;

ii) $(B_{t \wedge T_\mu}; t \geq 0)$ is uniformly integrable.

As a consequence of ii), we have :

$$B_{t \wedge T_\mu} = E[B_{T_\mu} | \mathcal{F}_{t \wedge T_\mu}] \geq -c.$$

Hence, if we denote $T_{-c} = \inf\{t \geq 0; B_t = -c\}$, we obtain $T_{-c} \geq T_\mu$ a.s., so that:

$$T_{-c} = T_\mu + \inf\{v \geq 0; B_{v+T_\mu} - B_{T_\mu} = -c - B_{T_\mu}\}.$$

Since the Brownian motion $(B_{v+T_\mu} - B_{T_\mu}; v \geq 0)$ is independent from $\mathcal{F}_{T_\mu}$, we may write:

$$T_{-c} \overset{(law)}{=} T_\mu + (c + B_{T_\mu})^2 T$$

where, on the right-hand side, T is independent from the pair (T_μ, B_{T_μ}).
To finish the proof, we remark that :

$$T_{-1} \stackrel{(law)}{=} T_{-c} + (1-c)^2 T'$$

where T' is again a stable $(\frac{1}{2})$ variable, independent from T_{-c}, so that we have finally obtained : $T \stackrel{(law)}{=} S + L^2 T$ where: $S = (1-c)^2 T' + T_\mu$; $L = c + B_{T_\mu}$.
If $E(S) < \infty$, then $X_\lambda := \frac{\lambda^2}{2} S + \lambda L$ $(\lambda \geq 0)$ is integrable. By Jensen's inequality,

$$\exp(-E(X_\lambda)) \leq E(\exp(-X_\lambda)),$$

which implies that $\lambda \leq \frac{\lambda^2}{2} E(S) + \lambda E(L)$. Dividing the inequality by λ and letting $\lambda \to 0$, we obtain $E(L) \geq 1$. □

Remark:

1) To illustrate the above construction, we recall the explicit construction of T_μ given by Azéma-Yor [**?**]:
if $S_t := \sup_{s \leq t} B_s$ and $\Psi_\mu(x) = \frac{1}{\mu([x,\infty[)} \int_{[x,\infty[} y \, d\mu(y)$, then the stopping time :

$$T_\mu = \inf\{t \geq 0; S_t \geq \Psi_\mu(B_t)\}$$

solves Skorohod's embedding problem relative to μ, i.e. $B_{T_\mu} \sim \mu$.

2) The last assertion in Theorem 1.1 admits no converse. Indeed, if the distribution μ of $L-1$ satisfies $\int x^2 \mu(dx) = \infty$, then $S = T_\mu$ given by the construction of Azéma-Yor satisfies $E(S) = \infty$. (see [**?**])

In the second part of this note, rather than trying to develop an extended "zoology" of pairs (S, L) such that (**??**) is satisfied, we concentrate on examples where S and L are independent. On the analytic side, we see, from (**??**), that this corresponds to the factorization of : $\lambda \to \exp(-\lambda)$ as:

$$\exp(-\lambda) = \varphi_1(\lambda)\varphi_2(\frac{\lambda^2}{2}), \tag{1.6}$$

where φ_1 and φ_2 are two Laplace transforms of probabilities on $\mathbb{R}_+$. In terms of S and L, $\varphi_1(\lambda) = E[\exp(-\lambda L)]$; $\varphi_2(\mu) = E[\exp(-\mu S)]$, $\lambda, \mu \geq 0$.

If, moreover, we assume that L is infinitely divisible, there is the following characterization.

Theorem 1.2 *Let $L \geq 0$ be infinitely divisible with Lévy exponent:*

$$\psi(\lambda) = \int_0^\infty \mu(dt)(1 - \exp(-\lambda t)).$$

In order that, on an adequate probability space, a variable S may be constructed such that:

i) S and L are independent;

*ii) (S, L) satisfies (**??**),*

it is necessary and sufficient that:

$$\psi(\lambda) \leq \lambda \tag{1.7}$$

or, equivalently,

$$\int_0^\infty t \; \mu(dt) \leq 1. \tag{1.8}$$

In this case, $E(L) = \int_0^\infty t \; \mu(dt)$ and S is also infinitely divisible.

Proof: a) We first assume that i) and ii) are satisfied. Thus we have:

$$\exp(-\lambda) = \exp(-\psi(\lambda)) E(\exp(-\frac{\lambda^2}{2} S)),$$

which immediately implies (**??**).
We now show that (**??**) and (**??**) are equivalent. Indeed, if (**??**) is satisfied, then:

$$1 \geq \frac{\psi(\lambda)}{\lambda} = \int_0^\infty \mu(dt) \int_0^t ds \exp(-\lambda s);$$

Letting λ decrease to 0, we obtain (**??**).
Conversely, if (**??**) is satisfied, since $1 - \exp(-\lambda t) \leq \lambda t$, we immediately obtain (**??**).
The differentiation of the Laplace transform of L with respect to λ gives

$$E(L) = \int_0^\infty t \; \mu(dt).$$

b) Conversely, we now assume that (**??**) (or equivalently (**??**)) is satisfied, then we shall show that there exists a Lévy measure $\rho(dt)$ on $\mathbb{R}_+$ such that:

$$\lambda - \psi(\lambda) = \int_0^\infty \rho(dt)(1 - \exp(-\frac{\lambda^2}{2} t)) \tag{1.9}$$

which, a fortiori, proves the desired result.
Assuming, for a moment, the existence of ρ, we obtain, after taking derivatives on both sides, and dividing by λ :

$$\frac{1}{\lambda} = \frac{1}{\lambda}\int_0^\infty \mu(dt)\ t\exp(-\lambda t) + \int_0^\infty \rho(dt)\ t\exp(-\frac{\lambda^2}{2}t). \tag{1.10}$$

Since both $\frac{1}{\lambda}$ and $\frac{1}{\lambda}\int_0^\infty \mu(dt)t\exp(-\lambda t)$ are Laplace transforms in $\frac{\lambda^2}{2}$, we obtain, from the injectivity of the Laplace transform :

$$\frac{1}{\sqrt{2\pi t}} = \frac{1}{\sqrt{2\pi t}}\int_0^\infty \mu(ds)\ s\exp(-\frac{s^2}{2t})\ +\ t\,(\rho(dt)/dt)\,, \tag{1.11}$$

showing at the same time, that $\rho(dt)$ must be absolutely continuous.
Now, we start working backwards : indeed, the condition (**??**) implies that :

$$\frac{1}{\sqrt{2\pi t}}\left(1 - \int_0^\infty \mu(ds)s\exp(-\frac{s^2}{2t})\right)$$

is positive, and then $\rho(dt)$ defined from (**??**) is indeed a Lévy measure, i.e. : it satisfies $\int_0^\infty \rho(dt)(t\wedge 1) < \infty$, since this is the case with the left hand side of (**??**) divided by t; more precisely :

$$\int_0^\infty \frac{dt}{t^{3/2}}(t\wedge 1) < \infty. \qquad \square$$

Remark: From this theorem, we can give an explicit construction of a pair (S, L) satisfying the condition of Theorem 1.2 with $E(L) = 1$ and $E(S) = \infty$. Indeed, take $\mu(dt) = (\alpha+1)\dfrac{dt}{t^{\alpha+1}}1_{(t\geq 1)}$, $1 < \alpha \leq 2$, then the Lévy measure ρ of S given by (**??**) satisfies $\int_0^\infty t\rho(dt) = \infty$ which implies $E(S) = \infty$.

2 Examples of pairs (S, L).

In the following examples,$(B_t, t\geq 0)$ denotes a Brownian motion and

$$T = \inf\{t\geq 0; B_t = 1\}.$$

The following remarks yield an important class of solutions to (**??**):

1) By the scaling property of T, if (S,L) is a solution of (**??**) with two independent variables S and L, then for all $n \in I\!N^*$, $\left(\frac{S^{*n}}{n^2}, \frac{L^{*n}}{n}\right)$ is still a solution to (**??**) where ^{*n} denotes the convolution of order n. Moreover, if S and L are infinitelty divisible, we can replace $n \in I\!N^*$ by $\mu \in I\!R_+$.

2) to any stopping time S of the Brownian filtration satisfying $S \leq T$, we can associate an affine decomposition of the stable $(\frac{1}{2})$ variable. Indeed,

$$T = S + \inf\{t \geq 0, B_{t+S} - B_S = 1 - B_S\}$$

and $T \overset{(law)}{=} S + (1-B_S)^2 T$ where on the right hand side, T is independent of (S, B_S).
We are now looking for such stopping times S such that S and B_S are independent.

Example 1: This is the case for $S = T_a^* = \inf\{t \geq 0; |B_t| = a\}$, $a \leq 1$. In this case, the factorization (**??**) corresponds to :

$$\varphi_1(\lambda) = \exp(-\lambda)(\cosh(\lambda a));\ \varphi_2(\frac{\lambda^2}{2}) = \frac{1}{\cosh(\lambda a)}.$$

Example 2: Let $S_t = \sup_{s \leq t} B_s$. According to Pitman's theorem, $(R_t := 2S_t - B_t, t \geq 0)$ is a Bessel process of dimension 3, and conditionally to $\mathcal{R}_t = \sigma(R_s; s \leq t)$, the distribution of B_t is uniform on $[-R_t, R_t]$.
Thus, if $S = \inf\{t \geq 0; 2S_t - B_t = a\}$ $(a \leq 1)$, B_S is uniform on $[-a, a]$ and independent of S.

The decomposition (*) $T = S + (1-B_S)^2 T$ leads to a decomposition (**??**) with S and L independent. From the analytic side,

$$\varphi_1(\lambda) = \frac{\sinh(\lambda a)\exp(-\lambda)}{\lambda a};\ \varphi_2(\frac{\lambda^2}{2}) = \frac{\lambda a}{\sinh(\lambda a)}.$$

Remark: We can also interpret the decomposition (*) with the help of Williams decomposition of the Brownian path $(B_t, t \leq T)$. Let

$$\Sigma = \sup\{t \leq T, B_t = 0\},$$

then, $T = (T - \Sigma) + \Sigma$ where $T - \Sigma$ and Σ are independent; the first time is distributed as $T^{(3)}$ the first hitting time of 1 by a BES(3); the second one as $(2U)^2T$ where U is uniform on [0,1], independent of T. Thus, we recover equation (*) for $a = 1$.

Example 3: In the previous example, we gave a decomposition of T with $S \stackrel{(law)}{=} T_a^{(3)}$. This suggests to look for an affine decomposition with $S = T_a^{(d)}$ the first hitting time of a by a BES(d).
Let $d = 2(\nu + 1) > 1$ and consider R_{d-1} a Bessel process of dimension $d - 1$, independent of (B_t), defined on the same probability space. Then R_d defined by

$$R_d^2(t) = R_{d-1}^2(t) + B^2(t)$$

is a BES(d) process. Let

$$T^{(d)} = \inf\{t \geq 0; R_d(t) = 1\} < T,$$

then

$$T = T^{(d)} + (1 - B_{T^{(d)}})^2 T \tag{2.1}$$

where on the RHS of (**??**), $(T^{(d)}, B_{T^{(d)}})$ is independent of T.
Moreover, according to the strong intertwining relation established in [**?**, Theorem 3.1],

$$E[f(B_t^2)/\mathcal{G}_t] = \Lambda_{1/2,(d-1)/2} f(R_d^2(t)) \tag{2.2}$$

where $\mathcal{G}_t = \sigma\{R_d(s); s \leq t\}$, and

$$\Lambda_{a,b} f(y) = E[f(yZ)]$$

where Z denotes a beta(a, b) variable. The identity (**??**) extends when we replace t by any $\mathcal{G}_t$ stopping time S. This implies that $B_{T^{(d)}}^2$ is independent of $\mathcal{G}_{T^{(d)}}$ (and therefore of $T^{(d)}$) and is distributed as $Z_{1/2,\nu+1/2}$.
From this, we easily deduce that

$$(1 - B_{T^{(d)}}) \stackrel{(law)}{=} 2X_\nu$$

where X_ν is a beta$(\nu + 1/2, \nu + 1/2)$ variable, and (**??**) becomes :

$$T \stackrel{(law)}{=} T^{(d)} + (2X_\nu)^2 T$$

with independence of the three variables on the right-hand side.
The corresponding factorization is

$$\exp(-\lambda) = \left(C_\nu \frac{\lambda^\nu}{I_\nu(\lambda)} \right) \frac{1}{C_\nu} \lambda^{-\nu} I_\nu(\lambda) \exp(-\lambda) \tag{2.3}$$

with $C_\nu = \frac{1}{2^\nu \Gamma(\nu+1)}$ and $\frac{1}{C_\nu}\lambda^{-\nu} I_\nu(\lambda)\exp(-\lambda) = E(\exp(-2\lambda X_\nu))$.

Example 4: It is now tempting to look for a decomposition (**??**) with $S = L^{(d)} = \sup\{t \geq 0, R_d(t) = 1\}$ where R_d is $BES(d)$ with $d > 3$, although $L^{(d)}$ is not a stopping time, at least in the natural filtration of the Bessel process.
From the analytic point of view, we replace $C_\nu \frac{\lambda^\nu}{I_\nu(\lambda)}$ in the factorization (**??**) by $C'_\nu \lambda^\nu K_\nu(\lambda)$ with $C'_\nu = \frac{2^{1-\nu}}{\Gamma(\nu)}$ and the problem is the following :
is the function

$$\lambda \mapsto \varphi_1^{(\nu)}(\lambda) := \frac{\exp(-\lambda)}{c'_\nu \lambda^\nu K_\nu(\lambda)} \tag{2.4}$$

the Laplace transform of a probability on $\mathbb{R}_+$?
The answer is positive for $\nu = 3/2$ ($d = 5$). In this case, $\varphi_1(\lambda) = \frac{1}{1+\lambda}$ is the Laplace transform of the exponential variable, with mean 1. This leads to the decomposition

$$T \overset{(law)}{=} L^{(5)} + \mathbf{e}^2 T \tag{2.5}$$

where $L^{(5)}$, T and $\mathbf{e}$ are independent. On the analytical side,

$$E\left(\exp(-\frac{\lambda^2}{2}T)\right) = \exp(-\lambda) \equiv ((1+\lambda)\exp(-\lambda))\,\frac{1}{1+\lambda}$$

When $\nu = n + 1/2$, $n \in \mathbb{N}$, the function $K_{n+1/2}$ can be expressed as (see [**?**]):

$$K_{n+1/2}(z) = (\frac{1}{z})^{n+1/2}\sqrt{\frac{\pi}{2}}\exp(-z)P_n(z)$$

where P_n is the Bessel polynomial given by:

$$P_n(z) = \sum_{j=0}^{n}(\frac{1}{2})^j \frac{(n+j)!}{j!(n-j)!} z^{n-j}.$$

We refer to Ismail-Kelker [**?**] where quotients of Bessel polynomials arose in connection with the problem of the infinite divisibility of the Student distribution.
In this particular case, the above question boils down to knowing whether $\frac{a_n}{P_n(\lambda)}$ is the Laplace transform in λ of a probability on $\mathbb{R}_+$, with $a_n = \sqrt{\frac{2}{\pi}}\Gamma(n+1/2)2^{n-1/2}$.
We recall the following result:

Proposition 2.1 *([?]) If the function* $\Phi(\lambda) = \dfrac{1}{c_0 + c_1\lambda + \ldots + c_n\lambda^n}$, $c_i \in \mathbb{R}$, $c_n \neq 0$ *is the Laplace transform of a probability on* $\mathbb{R}_+$*, then:*

i) $c_0 = 1$,

ii) the polynomial $P(z) = 1/\Phi(z)$ *does not have any root in* $i\mathbb{R}$,

iii) if $a \pm ib$ $(a \neq 0,\ b \neq 0)$ *are two conjugate roots of* P*, then* P *has at least a real root* c *satisfying :* $sgn(a) = sgn(c)$ *and* $|c| \leq |a|$.

If $n \leq 4$*, the conditions i), ii) and iii) are sufficient.*

It is known (see [?, p. 194]) that the zeros of the Bessel polynomials are distinct, that they all lie in the left half plane and that there is only one real zero for odd n and none for even n. In particular, for even n, the condition iii) is not satisfied.
For $n = 3$, $\frac{P_3(\lambda)}{a_3} = \frac{1}{15}(\lambda^3 + 6\lambda^2 + 15\lambda + 15)$ and we can show that the last condition in iii) is not satisfied.

A natural question is to find a probabilistic interpretation of equation (??) (similar to Eq (??)).
Let $d \in \mathbb{N}$, $d > 3$, it is tempting, following Example 3, to consider $B_t^{(d)}$ a d dimensional Brownian motion, with components $B_i(t)$, which we decompose into $(B_t^{(3)}, B_t^{(d-3)})$. Now, since $T \overset{(law)}{=} L^{(3)}$, and that, obviously, $L^{(3)} > L^{(d)}$, we can write :

$$L^{(3)} = L^{(d)} + \sup\{v \geq 0; |B^{(3)}_{L^{(d)}+v}| = 1\} := L^{(d)} + \Lambda^{3,d}. \tag{2.6}$$

We shall see that this decomposition provides a decomposition of $L^{(3)}$ in two independent variables.
First, we recall that the variable $B^{(d)}_{L^{(d)}}$ is uniformly distributed on the unit sphere S^{d-1} ($\subset \mathbb{R}^d$) and is independent from $L^{(d)}$. Now, we can write the decomposition of $B^{(d)}$ in the filtration $(\mathcal{F}_{L^{(d)}+t}; t \geq 0)$ using the progressive enlargement of filtration after time $L^{(d)}$ (see [?]). According to [?, Theorem 12.5], for any $\mathcal{F}_t$ local martingale X,

$$X_{L^{(d)}+t} = X_{L^{(d)}} + \tilde{X}_t - \int_{L^{(d)}}^{L^{(d)}+t} \frac{d\langle X, Z\rangle_s}{1 - Z_s} \tag{2.7}$$

where (Z_t) is the $(\mathcal{F}_t)$ supermartingale: $Z_t = P(L^{(d)} > t|\mathcal{F}_t)$ and $\tilde{X}$ is a $(\mathcal{F}_{L^{(d)}+t}; t \geq 0)$ local martingale.
Now, it is easily shown that $Z_t = 1 \wedge \left(\frac{1}{|B_t^{(d)}|}\right)^{d-2}$ (see [?, (12.2.2)]), and that for $t \geq L^{(d)}$,

$$Z_t = 1 - (d-2) \int_{L^{(d)}}^t \frac{d\beta_u}{|B_u^{(d)}|^{d-1}}$$

where (β_t) is the $(\mathcal{F}_t)$ Brownian motion

$$\beta_t = \int_0^t \sum_{i=1}^d \frac{B_i(u)\ dB_i(u)}{|B^{(d)}(u)|}.$$

The decomposition (**??**) for $X = B^{(d)}$ gives:

$$\hat{B}_t := B^{(d)}_{L^{(d)}+t} = \hat{B}_0 + \tilde{B}_t + (d-2)\int_0^t \frac{\hat{B}_u du}{|\hat{B}_u|^2(|\hat{B}_u|^{d-2}-1)}$$

where $\hat{B}_0$ is uniform on S^{d-1} and independent of $L^{(d)}$ and $\tilde{B}$ is a d dimensional $(\mathcal{F}_{L^{(d)}+t}; t \geq 0)$ Brownian motion.
This confirms that the process $\hat{B}$ is independent of $L^{(d)}$ and in the decomposition (**??**), the two variables on the right-hand side are independent.

From (**??**), it follows that:

$$\Lambda^{3,5} \stackrel{(law)}{=} \mathbf{e}^2 T \stackrel{(law)}{=} T_{\mathbf{e}}, \tag{2.8}$$

but a priori, we do not know of any such representation for $\Lambda^{3,d}$, when $d = 3+m, m \in I\!N^*$. In particular, if $d = 4k+3, k \in I\!N^*$, $\Lambda^{3,d}$ is not distributed as L^2T for any positive variable L independent of T, since $\varphi_1^{(2k+1/2)}$, defined by (**??**), is not the Laplace transform of a probability on $I\!R_+$ (see the discussion after Proposition 2.1). Moreover, we have not found any pathwise explanation to the identity (**??**), but we mention that it may be related to the general result (due to Azéma) that if A_∞ denotes the terminal variable of (A_t), the dual predictable projection of $1_{(\Sigma \leq t)}$, where Σ is the end of a predictable set such that Σ avoids $(\mathcal{G}_t)$ stopping times, then A_∞ is exponentially distributed.
We conjecture that there is a critical value ν_c for which $\varphi_1^{(\nu)}$ is a Laplace transform for ν smaller than ν_c but is not for ν greater than ν_c.

Example 5: We come back to Eq (**??**) : $T \overset{(law)}{=} L^{(5)} + \mathbf{e}^2 T$, which may be extended as follows. The factorization corresponding to (**??**) :

$$\exp(-\lambda) = ((1+\lambda)\exp(-\lambda))\ \frac{1}{1+\lambda}$$

can be generalized as

$$\exp(-\lambda) = \left((1+\frac{\lambda}{\alpha})^\alpha \exp(-\lambda)\right)\ \frac{1}{(1+\frac{\lambda}{\alpha})^\alpha}$$

for any $\alpha > 0$, and

$$\frac{1}{(1+\frac{\lambda}{\alpha})^\alpha} = E(\exp(-\lambda\frac{\mathbf{e}_\alpha}{\alpha}))$$

where $\mathbf{e}_\alpha$ denotes a Gamma variable of parameter α. Let (the law of) S_α be defined by

$$E(\exp(-(\lambda^2/2)\ S_\alpha)) = (1+\frac{\lambda}{\alpha})^\alpha \exp(-\lambda), \tag{2.9}$$

then,

$$T \overset{(law)}{=} S_\alpha + \left(\frac{\mathbf{e}_\alpha}{\alpha}\right)^2 T.$$

The existence of a distribution satisfying (**??**) follows from Theorem 1.2, since, in this case, $\psi(\lambda) = \alpha\ln(1+\frac{\lambda}{\alpha})$ satisfies $\psi(\lambda) \leq \lambda$. Moreover, the Lévy measure ρ_α associated with S_α is given by (**??**) that is :

$$\rho_\alpha(dt) = \frac{1}{\sqrt{2\pi t^3}}\left(1 - \int_0^\infty \mu(ds)\ s\exp(-\frac{s^2}{2t})\right) dt$$

where $\mu(dt)$ is the Lévy measure of $\frac{\mathbf{e}_\alpha}{\alpha}$ i.e.:

$$\mu(dt) = \alpha\frac{\exp(-\alpha t)}{t}dt.$$

An easy computation leads to :

$$\rho_\alpha(dt) = \frac{dt}{\sqrt{2\pi t^3}} g(\alpha\sqrt{t}) \tag{2.10}$$

where

$$g(\xi) = \int_0^\infty dy\ y\exp(-\frac{y^2}{2} - \xi y).$$

We notice that $S_\alpha \overset{(law)}{=} \frac{(L^{(5)})^{*\alpha}}{\alpha^2}$.

3 Further examples using infinite divisibility

Example 5 led us to a large class of extensions. Indeed, from that Example, it is natural to consider the factorisation:

$$\exp(-\lambda) = \frac{1}{(1+\frac{\lambda}{b})^{\alpha}} \left\{ (1+\frac{\lambda}{b})^{\alpha} \exp(-\lambda) \right\} \tag{3.1}$$

$$\equiv \varphi_1(\lambda)\varphi_2(\frac{\lambda^2}{2}), \tag{3.2}$$

and to ask for which values of b, φ_2 is the Laplace transform of a distribution on $I\!R^+$ (see (**??**) above). It is easily shown that this is satisfied for $b \geq \alpha$. We now extend this remark as follows:

Theorem 3.1 *Let $\nu(da)$ be a positive measure with compact support on $I\!R_+$, and total mass $\theta < \infty$. We consider the decomposition:*

$$\exp(-\lambda) = \varphi_1(\lambda)\varphi_2(\frac{\lambda^2}{2}),$$

where

$$\varphi_1(\lambda) = \exp\left(-\int \nu(da)(\ln(1+\lambda a))\right) \tag{3.3}$$

$$\varphi_2(\frac{\lambda^2}{2}) = \exp\left(-\int \nu(da)(\frac{\lambda}{\theta} - \ln(1+\lambda a))\right). \tag{3.4}$$

Then,

a) φ_1 is the Laplace transform of $\int_0^\infty a \; d\left(\gamma_{\nu([0,a])}\right)$ where $(\gamma_t; t \geq 0)$ denotes the standard gamma process;

b) φ_2 is the Laplace transform of a positive measure on $I\!R_+$, as soon as $\int a \; \nu(da) \leq 1$.

Comment 1: For a number of results about the gamma process $(\gamma_t; t \geq 0)$, see e.g., Vershik-Yor [**?**], Tsilevich-Vershik-Yor [**?**], Diaconis-Kemperman [**?**].
Proof:

a) The gamma process $(\gamma_t; t \geq 0)$ is the subordinator such that, for each $t \geq 0$, γ_t is distributed as gamma (t). As a consequence, it is easily shown that for any bounded Borel function $f : [0,\infty[\to \mathbb{R}_+$, the following formula holds:

$$E[\exp\left(-\lambda \int_0^\infty f(a)\ d(\gamma_{\nu([0,a])})\right)] = \exp\left(-\int \nu(da)\ln(1+\lambda f(a))\right),$$

which yields the first part of the theorem.

b) It remains to find under which condition on ν, there exists a Lévy measure $m_\nu(dt)$ on $\mathbb{R}_+$, such that:

$$\int \nu(da)[\frac{\lambda}{\theta} - \ln(1+\lambda a)] = \int m_\nu(dt)[1-\exp(-\frac{\lambda^2}{2}t)].$$

From Theorem 1.2, the existence of m_ν is equivalent to the condition:

$$\int \nu(da)\ln(1+\lambda a) \leq \lambda \ \text{ for all } \lambda \geq 0,$$

which is equivalent to: $\int \nu(da)\ a \leq 1$.
Moreover, from Example 5, we can explicitely compute m_ν. Replacing λ by $\frac{\lambda}{\theta}$, and a by $(a\theta)$, we may restrict the discussion to the case: $\theta = 1$.

$$\begin{aligned}\lambda - \ln(1+\lambda a) &= a(\lambda - \frac{1}{a}\ln(1+\lambda a)) + (1-a)\lambda \\ &= a\int_0^\infty \rho_{1/a}(dt)t\exp(-\frac{\lambda^2}{2}t) \ + \ (1-a)\int_0^\infty \rho_0(dt)t\exp(-\frac{\lambda^2}{2}t)\end{aligned}$$

where ρ_b is defined by (**??**). Thus,

$$m_\nu = \int \nu(da)a\rho_{1/a} \ + \left(\int \nu(da)(1-a)\right)\rho_0. \qquad \square$$

4 Some concluding remarks:

1) In Theorem 1.1, we have mentioned the explicit construction given in Azéma-Yor [**?**] to solve Skorohod's embedding problem. We ask ourselves whether the previous examples of Section 2 can be constructed with the help of this explicit construction. More precisely, we consider the affine decomposition :

$$T \overset{(law)}{=} S + (1-B_S)^2 T$$

for any stopping time $S \leq T$, with a prescribed distribution μ for B_S. The question is: does the construction of [?] provide a solution (S, B_S) with independence of the two components?
The answer is positive in the case of Example 2. This corresponds to μ the uniform distribution on $[-a, a]$.Then, $\Psi_\mu(x) = \frac{x+a}{2}$ and

$$S := \inf\{t \geq 0; S_t \geq \Psi_\mu(B_t)\} = \inf\{t \geq 0; 2S_t - B_t \geq a\}.$$

On the other hand, the construction of Azéma-Yor [?] provides an example of an affine decomposition (**??**) with $L = \mathbf{e}$ (as in Example 5) but with no independence property between S and L. Indeed, consider $\mu(dx) = \exp(-(x+1))1_{[-1,\infty[}(x)dx$ (see [?], 5.2.b). Then $T_\mu \leq T_{-1}$ and

$$T \overset{(law)}{=} T_{-1} \overset{(law)}{=} T_\mu + (1 + B_{T_\mu})^2 T,$$

with $1 + B_{T_\mu} \overset{(law)}{=} \mathbf{e}$. In this case , $T_\mu = \inf\{t \geq 0; S_t - B_t = 1\} \overset{(law)}{=} T_1^{(1)}$ the first hitting time of 1 by a BES(1), i.e.: a reflecting Brownian motion.

2) In close relation with Section 2, we mention the paper [?], where the authors are looking for stopping times S of the Brownian filtration such that S and B_S are independent. Recall that in Section 2 (Examples 1 and 2), we are looking for such a time satisfying moreover $S \leq T_1$ a.s..
We recall a general framework which gives such times. Let (B_t) be a $(\mathcal{F}_t)$ Brownian motion and $(\mathcal{G}_t)$ a subfiltration of $(\mathcal{F}_t)$. Let (Z_t) be a $(\mathcal{G}_t)$ Markov process such that there exists a Markov kernel satisfying:

$$\forall f \geq 0, \qquad E[f(B_t)|\mathcal{G}_t] = Kf(Z_t). \tag{4.1}$$

Then, if $T_a = \inf\{t \geq 0; Z_t = a\}$, T_a and B_{T_a} are independent, and the distribution of B_{T_a} is $K(a, dx)$.

Applications:

i) $(\mathcal{G}_t) = \sigma(2S_s - B_s; s \leq t)$. This is Example 2.

ii) Another example follows from Matsumoto-Yor [?]: $(\mathcal{G}_t) = \sigma(Z_s; s \leq t)$ with

$$Z_t = \exp(-B_t)\int_0^t \exp(2B_s)ds.$$

Then for $S_z = \inf\{t \geq 0; Z_t = z\}$, B_{S_z} and S_z are independent. Of course, in this example, we cannot compare S_z with T_1.

3) In the quoted work of Matsumoto-Yor [?], the study of Z involves a random quadratic equation :

$$A \overset{(law)}{=} zX + AX^2 \tag{4.2}$$

where $A \overset{(law)}{=} \frac{1}{2\Gamma_\mu}$ (Γ_μ is a Gamma(μ) variable), $z \geq 0$ and X is independent of A.
For $\mu = 1/2$, (??) admits a unique solution $X_z \overset{(law)}{=} I^{(-1/2)}_{\frac{1}{\sqrt{z}},\frac{1}{\sqrt{z}}}$ where $I^{(\mu)}_{a,b}$ denotes a generalized inverse Gaussian distribution (see [?]).
Since $T \overset{(law)}{=} \frac{1}{2\Gamma_{1/2}}$, the pair $(S, L) = (zX_z, X_z)$ with $X_z \overset{(law)}{=} I^{(-1/2)}_{\frac{1}{\sqrt{z}},\frac{1}{\sqrt{z}}}$ provides a solution to (??). The density of X_z is given by :

$$f(u) = \frac{1}{\sqrt{2\pi z u^3}} \exp\left(-(\frac{1}{2zu} + \frac{u}{2z}) + \frac{1}{z}\right).$$

5 Appendix

We come back to Eq (??):

$$L^{(3)} \overset{(law)}{=} L^{(5)} + \mathbf{e}^2 T$$

to connect it with the decomposition:

$$L^{(3)} = \int_0^{L^{(3)}} ds 1_{(R_s^{(3)} \leq 1)} + \int_0^{L^{(3)}} ds 1_{(R_s^{(3)} \geq 1)}.$$

In this equality, the two terms on the right-hand side are not independent. Nevertheless, we have:

$$\int_0^{L^{(3)}} ds 1_{(R_s^{(3)} \geq 1)} \overset{(law)}{=} \mathbf{e}^2 T.$$

Indeed, by time reversal,

$$\int_0^{L^{(3)}} ds 1_{(R_s^{(3)} \geq 1)} \overset{(law)}{=} \int_0^{T_1} ds 1_{(B_s \leq 0)}$$

where B is a Brownian motion.
Let $A_t^- = \int_0^t ds 1_{(B_s \leq 0)}$ and τ denotes the inverse of the local time $l^0_\cdot$ of B.

Now, the process $(A^-_{\tau(u)}, u \geq 0)$ is independent of $L^0_{T_1}$ and $A^-_{\tau(u)} \stackrel{(law)}{=} T_{\frac{1}{2}u}$ where T_a is the first hitting time of a by a Brownian motion (see [**?**, p. 232]). Therefore,

$$A^-_{T_1} = A^-_{\tau(L^0_{T_1})} \stackrel{(law)}{=} (\frac{1}{2}L^0_{T_1})^2 T_1 \stackrel{(law)}{=} \mathbf{e}^2 T.$$

This extends to dimension d as follows:

Proposition 5.1 *1) Let $d \in \mathbb{N}^*$, there exists a random variable $X^{(d)}$ such that:*

$$L^{(d)} \stackrel{(law)}{=} L^{(d+2)} + X^{(d)}$$

with independence of the two variables on the right-hand side.

2) The decomposition

$$L^{(d)} = \int_0^{L^{(d)}} ds 1_{(R^{(d)}_s \leq 1)} + \int_0^{L^{(d)}} ds 1_{(R^{(d)}_s \geq 1)}.$$

as the sum of two non independent variables satisfies:

$$\int_0^{L^{(d)}} ds 1_{(R^{(d)}_s \geq 1)} \stackrel{(law)}{=} X^{(d)} \tag{5.1}$$

Proof: 1) As in Example 4, we consider a $(d+2)$-dimensional Brownian motion $B^{(d+2)}$, which we decompose into $(B^{(d)}, B^{(2)})$. Then, the decomposition

$$L^{(d)} = L^{(d+2)} + \sup\{v \geq 0; |B^{(d)}_{L^{(d+2)}+v}| = 1\}$$

provides a decomposition of $L^{(d)}$ in two independent variables. It follows that:

$$E[\exp(-\frac{\lambda^2}{2} X^{(d)})] = 2\nu \frac{K_\nu(\lambda)}{\lambda K_{\nu+1}(\lambda)}.$$

2) (**??**) holds by identification of the Laplace transform of the two variables since from [**?**, (9.n)],

$$E\left[\exp\left(-\frac{\lambda^2}{2}\int_0^{L^{(d)}} ds 1_{(R^{(d)}_s \geq 1)}\right)\right] = 2\nu \frac{K_\nu(\lambda)}{\lambda K_{\nu+1}(\lambda)}.\square$$

References

[1] Azéma, J. et Yor, M.: Une solution simple au problème de Skorokhod. *Sém. Prob.* XIII ; *Lecture Notes in Math.* 721, Springer 1979, 90-115.

[2] Babillot, M., Bougerol, P. and Elie, L.: The random difference equation : $X_n = A_n X_{n-1} + B_n$ in the critical case. *Annals of Probability* **25** (1997) 478-493.

[3] Carmona, P., Petit, F. and Yor, M.: Beta-gamma random variables and intertwining relations between certain Markov processes. *Revista Matemàtica Ibero-Americana* **14-2** (1998) 311-367.

[4] Diaconis, P. and Kemperman, J. : Some new tools for Dirichlet priors. *Bayesian Statistics* **5** (1996) 97-106.

[5] Ismail, M.E.H. and Kelker, D.H. : The Bessel polynomials and the Student t distribution. *Siam J. on Math. Anal.* **7** (1976) 82-91.

[6] Lukacs, E. and Szasz, O. Certain Fourier transforms of distributions. *Can. J. Math.* **3** (1951) 140-144.

[7] Luke, Y.L. *The special functions and their approximations*, vol. 2, John Wiley, New York, 1966.

[8] Matsumoto, H. and Yor, M. Some extensions of Pitman's theorem involving exponential Brownian functionals via generalized inverse Gaussian distributions. *Prépublication n*° 475. Laboratoire de Probabilités de l'Université Paris VI (1998).

[9] Pitman, J. and Yor, M. Bessel processes and infinitely divisible laws. *In: Stochastic Integrals*, ed. D. Williams, Lect. Notes in Maths. 851, Springer 1981.

[10] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion.* 2nd edition. Springer, Berlin, 1994.

[11] Roynette, B., Vallois, P. et Yor, M.: On independence cases of time and position of the Brownian motion. In preparation.

[12] Tsilevich, N., Vershik, A. and Yor, M.: Quasi-invariance properties of the gamma and the Dirichlet processes. In preparation (February 1999).

[13] Vershik, A. and Yor, M. Multiplicativité du processus gamma et étude asymptotique des lois stables d'indice α, lorsque α tend vers 0. *Prépublication n°* 289. Laboratoire de Probabilités de l'Université Paris VI.

[14] Watson, G.N. A treatise on the theory of Bessel functions. *Cambridge University Press* (1966).

[15] Yor, M. *Some Aspects of Brownian Motion. Part II: Some recent martingale problems* Lectures in Mathematics. ETH Zürich, Birkhäuser, 1997.