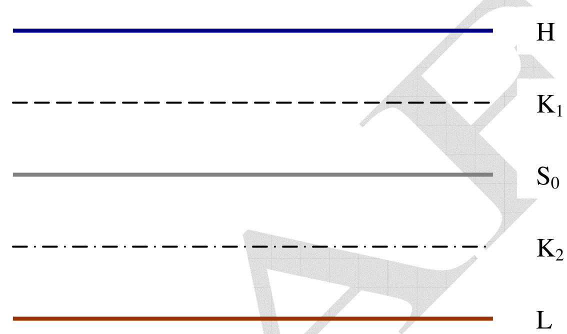
Barriers and Probabilities

We are often interested on the probability that the price of a stock will touch a price barrier or will be above a certain level after a given time period. Geometric Brownian Motion is one of the price models that helps us solve these kind of questions and more. We will start by first giving the answers in such questions. The answers will be given in the form of basic language code. We will then proceed to give a sketchy mathematical proof for these answers.

Let's start by defining the various price levels mentioned in the following formulas. If S_0 is the current price (at t=0) then K_1, H are two price levels above current price and K_2, L are two price levels bellow the current price. The following figure illustrates this relationship.



We will answer these questions

- `probability at time t the price is below a given barrier`
- `probability at time t the price is above a given barrier`
- `probability at some time until t max price is crossing above the H - barrier`
- `probability at any time until t max price is below the H - barrier`
- `probability at some time until t min price is crossing bellow the L - barrier`
- `probability at any time until t min price is above the L - barrier`
- `probability the final price is below the K1 - barrier and max price at some time until t is crossing above the H - barrier`
- `probability the final price is above the K2 - barrier and min price at some time until t is crossing below the L - barrier`

Basic Code Implementation

```
Function snorm(z As Double) as double
   'Cumulative normal distribution function (used in the calculations)
   Dim pi as double
   pi=3.14159265358979
   Dim a1 as double,a2 as double,a3 as double,a4 as double,a5 as
double,k as double,w as double
   a1 = 0.31938153
   a2 = -0.356563782
   a3 = 1.781477937
   a4 = -1.821255978
   a5 = 1.330274429
   If 0 > z Then w = -1 Else w = 1
   k = 1 / (1 + 0.2316419 * w * z)
   snorm = 0.5 + w * (0.5 - 1 / Sqr(2 * pi) * Exp(-z ^ 2 / 2) * (a1 * k
+ a2 * k ^ 2 + a3 * k ^ 3 + a4 * k ^ 4 + a5 * k ^ 5))
End Function


' ************ Inputs *************

' m drift (usually the risk free rate)
' s standard deviation (annualized volatility)
' t time (in years)
' H up barrier
' L down barrier
' K1,K2 barriers
' ST starting price


' ************ Formulas ************

' ProbPTBB probability at time t the price is below the barrier

' ProbPTUB probability at time t the price is above the barrier

' ProbMaxPTUB probability at some time until t max price is crossing
above the H - barrier

' ProbMaxPTBB probability at any time until t max price is below the H
- barrier

' ProbMinPTBB probability at some time until t min price is crossing
bellow the L - barrier

' ProbMinPTUB probability at any time until t min price is above the L
- barrier

' ProbFPBBMaxPTUB probability the final price is below the K1 - barrier
and max price at some time until t is crossing above the H - barrier

' ProbFPUBMinPTBB probability the final price is above the K2 - barrier
and min price at some time until t is crossing below the L - barrier
```

```
Function ProbPTBB(m as double, s as double, t as double, H as double,
ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbPTBB=snorm((log(H/ST)-MM*t)/(s*sqr(t)))
End Function

Function ProbPTUB(m as double, s as double, t as double, H as double,
ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbPTUB=1-snorm((log(H/ST)-MM*t)/(s*sqr(t)))
End Function

Function ProbMaxPTUB(m as double, s as double, t as double, H as
double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbMaxPTUB=snorm((-
log(H/ST)+MM*t)/(s*sqr(t)))+((H/ST)^(2*MM/s^2))*snorm((-log(H/ST)-
MM*t)/(s*sqr(t)))
End Function

Function ProbMaxPTBB(m as double, s as double, t as double, H as
double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbMaxPTBB=1-(snorm((-
log(H/ST)+MM*t)/(s*sqr(t)))+((H/ST)^(2*MM/s^2))*snorm((-log(H/ST)-
MM*t)/(s*sqr(t))))
End Function

Function ProbMinPTBB(m as double, s as double, t as double, L as
double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbMinPTBB=snorm((log(L/ST)-
MM*t)/(s*sqr(t)))+((L/ST)^(2*MM/s^2))*snorm((log(L/ST)+MM*t)/(s*sqr(t))
)
End Function

Function ProbMinPTUB(m as double, s as double, t as double, L as
double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbMinPTUB=1-(snorm((log(L/ST)-
MM*t)/(s*sqr(t)))+((L/ST)^(2*MM/s^2))*snorm((log(L/ST)+MM*t)/(s*sqr(t))
))
End Function

Function ProbFPBBMaxPTUB(m as double, s as double, t as double, H as
double, K1 as double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbFPBBMaxPTUB=((H/ST)^(2*MM/s^2))*snorm((log(K1/ST)-2*log(H/ST)-
MM*t)/(s*sqr(t)))
End Function
```

```
Function ProbFPUBMinPTBB(m as double, s as double, t as double, L as
double, K2 as double, ST as double) as double
   Dim MM as double
   MM=m-s^2/2
   ProbFPUBMinPTBB=((L/ST)^(2*MM/s^2))*snorm((-
log(K2/ST)+2*log(L/ST)+MM*t)/(s*sqr(t)))
End Function
```

Some Arithmetic Results

So	Starting price	100
T	Time	1 (year)
μ	Drift	0.02 (2% per year)
σ	Volatility	0.3 (30% per year)
H	H barrier	130
L	L barrier	75
K1	K1 barrier	110
K2	K2 barrier	90

Event	Formula Result	Monte Carlo Result*
Final price (after 1 year) above H barrier	16.91%	17.05%
Final price (after 1 year) below H barrier	83.09%	82.95%
Probability at some point until T the max Price is crossing above H	35.44%	34.63%
Probability at some point until T the min Price is crossing below L	36.50%	35.59%
Probability at some point until T the max Price is crossing above H and the final price is below K1	7.68%	7.15%
Probability at some point until T the min Price is crossing below L and the final price is above K2	5.80%	5.43%

* Monte Carlo results after 100 000 Iterations. Time T was divided into 1000 time steps.

Mathematical Proofs

$$dS/S = \mu \cdot dt + \sigma \cdot dz \Rightarrow \ln\left(\frac{S}{S_0}\right) \approx N\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot t, \sigma \cdot \sqrt{t}\right] \Rightarrow S(T) = S_o \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\cdot T + \sigma \cdot \varepsilon \cdot \sqrt{T}\right)$$

[1] $\Pr ob(S_T \le H) = N\left(\frac{\ln\left(\frac{H}{S_0}\right) - M \cdot T}{\sigma \cdot \sqrt{T}}\right)$, where $M = \mu - \frac{\sigma^2}{2}$

[2] Prob(S_T>H) = 1 – [1]

[3] Theorem (See Thorp[1]): For standard Brownian motion X(t)

$$\Pr ob(\sup[X(t) - (a \cdot t + b)] \ge 0, 0 \le t \le T) = N(-\alpha - \beta) + e^{-2ab} \cdot N(\alpha - \beta)$$

where $\alpha = a \cdot \sqrt{T}, \beta = \frac{b}{\sqrt{T}}$

[4] $$\Pr ob(\max[S(t)] \ge H) = N\left(\frac{M \cdot T - \ln\left(\frac{H}{S_0}\right)}{\sigma \cdot \sqrt{T}}\right) + \left(\frac{H}{S_0}\right)^{\frac{2M}{\sigma^2}} \cdot N\left(\frac{-M \cdot T - \ln\left(\frac{H}{S_0}\right)}{\sigma \cdot \sqrt{T}}\right)$$

proof: Use [3]

[5] Prob(max[S(t)]<H) = 1 – [4]

[6] $$\Pr ob(\min[S(t)] \le L) = N\left(\frac{-M \cdot T + \ln\left(\frac{L}{S_0}\right)}{\sigma \cdot \sqrt{T}}\right) + \left(\frac{L}{S_0}\right)^{\frac{2M}{\sigma^2}} \cdot N\left(\frac{M \cdot T + \ln\left(\frac{L}{S_0}\right)}{\sigma \cdot \sqrt{T}}\right)$$

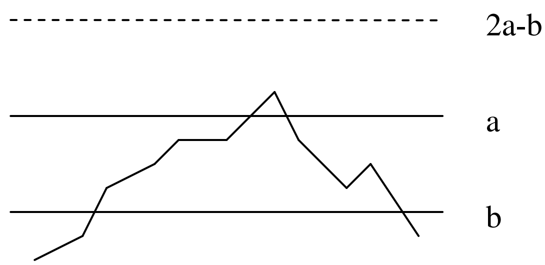
[7] Prob(min[S(t)]>L) = 1 – [6]

[8] For Brownian motion without drift Y(t):

$$\Pr ob(Y(T) \le b, \max[Y(t)] \ge a) = N\left(\frac{b - 2a}{\sigma \cdot \sqrt{T}}\right)$$

proof: Use the Reflection Principle (see chart bellow)

[1] Ed Thorp – "The Kelly criterion in blackjack, sports betting, and the stock market", 10th International Conference on Gambling and Risk Taking.



Hence for the Geometric Brownian Motion with Drift the formula becomes:

$$\Pr ob\big(S(T) \le K_1, \max[S(t)] \ge H\big) = \left(\frac{H}{S_0}\right)^{\frac{2M}{\sigma^2}} \cdot N\left(\frac{\ln\left(\frac{K_1}{S_0}\right) - 2\cdot\ln\left(\frac{H}{S_0}\right) - M\cdot T}{\sigma\cdot\sqrt{T}}\right)$$

where $\left(\frac{H}{S_0}\right)^{\frac{2M}{\sigma^2}}$ is the correction for the reflection in the barrier and $M \bullet T$ is the correction for the drift.

[9] $$\Pr ob\big(S(T) \ge K_2, \min[S(t)] \le L\big) = \left(\frac{L}{S_0}\right)^{\frac{2M}{\sigma^2}} \cdot N\left(\frac{-\ln\left(\frac{K_2}{S_0}\right) + 2\cdot\ln\left(\frac{L}{S_0}\right) + M\cdot T}{\sigma\cdot\sqrt{T}}\right)$$

[10] Using P(AB’) = P(A) – P(AB) we can calculate the following:

i. $Prob(S(T)\le K_1,max[S(t)]<H) = Prob(S(T)\le K_1) - Prob(S(T)\le K_1,max[S(t)]\ge H)$

ii. $Prob(S(T)\ge K_2,min[S(t)]>L) = Prob(S(T)\ge K_2) - Prob(S(T)\ge K_2,min[S(t)]\le L)$

iii. $Prob(S(T)>K_1,max[S(t)]\ge H) = Prob(max[S(t)]\ge H) - Prob(S(T)\le K_1,max[S(t)]\ge H)$

iv. $Prob(S(T)>K_1,max[S(t)]<H) = Prob(max[S(t)]<H) - Prob(S(T)\le K_1,max[S(t)]<H)$

v. $Prob(S(T)<K_2,min[S(t)]\le L) = Prob(min[S(t)]\le L) - Prob(S(T)\ge K_2,min[S(t)]\le L)$

vi. $Prob(S(T)<K_2,min[S(t)]>L) = Prob(min[S(t)]>L) - Prob(S(T)\ge K_2,min[S(t)]>L)$