WEIGHTED PARTIALLY ORDERD SETS OF FINITE TYPE

OLENA DROZD-KOROLEVA

Representations of posets (partially ordered sets) were introduced in [9]. In [7, 8] a criterion was given for a poset to be *representation finite*, i.e. having only finitely many indecomposable representations (up to isomorphism), and all indecomposable representations of posets of finite type were described. Further, in [4] Coxeter transformations were constructed for representations of posets, following the framework of [1]. It implied another criterion for a poset to be representation finite, not involving explicit calculations, but using the Tits quadratic form, also analogous to that of [1]. Note that this paper did not give all reflections, corresponding to the Tits form. They were constructed in [6], using a generalization of representations of posets, namely, representations of *bisected posets*.

Note that all these matrix problems are "split," i.e. do not involve extensions of the basic field. Some cases, when such extensions arise, were considered by Dlab and Ringel [2, 3]. The problems considered in [3] generalize representations of posets, though this generalization seems insufficient, especially when compared with [2].

Our aim is to present a more adequate generalization of representations of posets, which involves field extensions (even non-commutative), to construct the corresponding reflection functors and thus to obtain a criterion of representation finiteness, as well as a description of indecomposable representations in representation finite case. We call the arising problems *representations of weighed bisected posets.* They seem to be the most natural generalization of representations of posets allowing these constructions. By the way, even in "split" case they include the so called *Schurian vector space categories* (though nothing new arises in representation finite split case).

Since most proofs are quite similar to those of [6], we mainly only sketch them, though we give the details of all constructions, since they are not so evident.

1. Definitions and the Main Theorem

Recall [6] that a *bisected poset* is a poset $\mathbf{S}$ with a fixed partition $\mathbf{S} = \mathbf{S}^- \cup \mathbf{S}^+$ ($\mathbf{S}^- \cap \mathbf{S}^+ = \emptyset$) such that if $i \in \mathbf{S}^-$ and $j < i$, also $j \in \mathbf{S}^-$. We introduce a new symbol $0 \notin \mathbf{S}$ and set $\widehat{\mathbf{S}} = \mathbf{S} \cup \{\,0\,\}$, $\widehat{\mathbf{S}}^+ = \mathbf{S}^+ \cup \{\,0\,\}$, $\widehat{\mathbf{S}}^- = \mathbf{S}^- \cup \{\,0\,\}$. It is convenient, and we always do so, to set $0 < i$ for $i \in \mathbf{S}^-$ and $i < 0$ for $i \in \mathbf{S}^+$. Note that $<$ is an order on $\widehat{\mathbf{S}}^-$ and on $\widehat{\mathbf{S}}^+$, but not an order on $\widehat{\mathbf{S}}$. We write

- $i \lessdot j$ if $i < j$ and either both $i, j \in \widehat{\mathbf{S}}^-$ or both $i, j \in \widehat{\mathbf{S}}^+$;
- $i \ll j$ if $i < j$, $i \in \mathbf{S}^-$, $j \in \mathbf{S}^+$;
- $i \lessgtr j$ if $i < j$ or $j < i$ for $i, j \in \mathbf{S}$.

Let $\Bbbk$ be a fixed field (basic field). We consider finite dimensional skewfields (division algebras) over $\Bbbk$ and *finite dimensional bimodules* over such skewfields. If V is a $\mathbf{K}$-$\mathbf{L}$-bimodule and W is a $\mathbf{L}$-$\mathbf{F}$-bimodule, we write VW for the $\mathbf{K}$-$\mathbf{F}$-bimodule $V \otimes_{\mathbf{L}} W$. We also set $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and naturally identify it with $\mathrm{Hom}_{\mathbf{K}}(V, \mathbf{K})$ and with $\mathrm{Hom}_{\mathbf{L}}(V, \mathbf{L})$ as $\mathbf{L}$-$\mathbf{K}$-bimodules. We also use the natural isomorphisms

$$\text{(1)} \quad \mathrm{Hom}_{\mathbf{K}\text{-}\mathbf{F}}(UV, W) \simeq \mathrm{Hom}_{\mathbf{K}\text{-}\mathbf{L}}(U, WV^*) \simeq \mathrm{Hom}_{\mathbf{L}\text{-}\mathbf{F}}(V, U^*W) \simeq \\ \simeq \mathrm{Hom}_{\mathbf{F}\text{-}\mathbf{L}}(W^*U, V^*) \simeq \mathrm{Hom}_{\mathbf{L}\text{-}\mathbf{K}}(VW^*, U^*),$$

where U, V, W are, respectively, $\mathbf{K}$-$\mathbf{L}$-bimodule, $\mathbf{L}$-$\mathbf{F}$-bimodule and $\mathbf{K}$-$\mathbf{F}$-bimodule, as well as the duality isomorphism $V \simeq V^{**}$. If a map f belongs to one of these spaces, we usually denote by $\widetilde{f}$ its image in another one under the corresponding isomorphism.

Definition 1.1. A *weighted bisected poset*, or WBS, consists of:

- A finite poset $\mathbf{S} = \mathbf{S}^- \cup \mathbf{S}^+$. We
- a map $i \mapsto \mathbf{K}(i)$, where $i \in \widehat{\mathbf{S}}$ and $\mathbf{K}(i)$ is a finite dimensional skewfield over $\Bbbk$;
- a set of finite dimensional $\mathbf{K}(i)$-$\mathbf{K}(j)$-bimodules $V(ij)$, where $i, j \in \widehat{\mathbf{S}}$ and either $j \lessdot i$ or $i \ll j$;
- a set of $\mathbf{K}(i)$-$\mathbf{K}(j)$-linear maps $\mu(ikj) : V(ik)V(kj) \to V(ij)$ given for any triple $i, j, k \in \widehat{\mathbf{S}}$ such that all these bimodules are defined. We write uv for $\mu(ikj)(u \otimes v)$.

These maps must satisfy the following conditions:

1. "associativity": $\mu(ilj)(\mu(ikl) \otimes 1) = \mu(ikj)(1 \otimes \mu(klj))$ as soon as these maps are defined (it means that $(uv)w = u(vw)$);
2. "non-degeneracy":
   - if $j < i$, $i, j \in \mathbf{S}^-$ and $v \in V(ij)$, $v \neq 0$, there is an element $u \in V(j0)$ such that $vu \neq 0$;
   - if $j < i$, $i, j \in \mathbf{S}^+$ and $v \in V(ij)$, $v \neq 0$, there is an element $u \in V(0i)$ such that $uv \neq 0$;
   - if $j \ll i$, the map $\mu(j0i)$ is surjective.

We often write "a WBS $\mathbf{S}$" not mentioning the ingredients $\mathbf{S}^{\pm}$, $\mathbf{K}(i)$, $V(ij)$ and $\mu(ikj)$.

Definition 1.2. (1) A *representation* (M, f) of a WBS $\mathbf{S}$ consists of:

- finite dimensional $\mathbf{K}(i)$-vector spaces $M(i)$ given for each $i \in \widehat{\mathbf{S}}$;

- $\mathbf{K}(i)$-linear maps $f(i) : M(i) \to V(i0)M(0)$ given for each $i \in \mathbf{S}^-$;
- $\mathbf{K}(0)$-linear maps $f(i) : M(0) \to V(0i)M(i)$ given for each $i \in \mathbf{S}^+$,

such that the product

$$M(i) \xrightarrow{f(i)} V(i0)M(0) \xrightarrow{1\otimes f(j)} V(i0)V(0j)M(j) \xrightarrow{\mu(i0j)\otimes 1} V(ij)M(j)$$

is zero for every pair $i \ll j$. Again, we often write "a representation M" not mentioning f.

(2) A *morphism* $\phi : (M, f) \to (M, g)$ is a set of $\mathbf{K}(i)$-linear maps

$$\phi(i) : M(i) \to N(i) \quad \text{for all} \ \ i \in \hat{S},$$
$$\phi(ji) : M(i) \to V(ij)N(j) \quad \text{for } j \lessdot i,$$

that satisfy the following conditions:

$$g(i)\phi(0) = (1 \otimes \phi(i))f(i) + \sum_{i<j} (\mu(0ji) \otimes 1)(1 \otimes \phi(ij))f(j)$$

for $i \in S^+$ and

$$g(i)\phi(i) = (1 \otimes \phi(0))f(i) + \sum_{j<i} (\mu(ij0) \otimes 1)(1 \otimes g(j))\phi(ji)$$

for $i \in S^-$.

We denote by $\mathrm{Hom}_{\mathbf{S}}(M, N)$ the set of such morphisms.

Remark. If all skewfields $\mathbf{K}(i)$ as well as all bimodules $V(ij)$ coincide with the basic field $\Bbbk$ and all maps $\mu(ikj)$ are identities, these definitions coincide with the definitions of representations of bisected posets from [6]. If all $\mathbf{K}(i) = \Bbbk$ but not necessarily $V(ij) = \Bbbk$, we get a slight generalization of subspace categories of Schurian vector space categories [11]. Note that in the latter case the problem is never representation finite.

Representations of a WBS $\mathbf{S}$ and their morphisms form a $\Bbbk$-linear, fully additive category $\operatorname{rep}\mathbf{S}$. The unit morphism id_M in this category is such that $\mathrm{id}_M(i) = \mathrm{id}_{M(i)}$ for each i and all $\mathrm{id}_M(ij) = 0$. Since all spaces $\mathrm{Hom}_{\mathbf{S}}(M, N)$ are finite dimensional, it is a *Krull–Schmidt category*, i.e. every representation uniquely decomposes into a direct sum of indecomposable ones.

Definition 1.3. We call a WBS $\mathbf{S}$ *representation finite* if it only has finitely many non-isomorphic indecomposable representations. Otherwise we call it *representation infinite*.

We are going to find a criterion for a WBS to be representation finite and to describe indecomposable representations in representation finite case. To do it, just as in [1, 2, 4, 6], we use the *Tits form* and *reflection functors*.

Definition 1.4. For a WBS $\mathbf{S}$ we set $d_i = \dim_{\Bbbk} \mathbf{K}(i)$, $d_{ij} = d_{ji} = \dim_{\Bbbk} V(ij)$, consider the real vector space $\mathbb{R}^{\widehat{\mathbf{S}}}$ of functions $\mathbf{x} : \widehat{\mathbf{S}} \to \mathbb{R}$ and define the *Tits form* $\mathrm{Q}_{\mathbf{S}}$ as the quadratic form on the space $\mathbb{R}^{\widehat{\mathbf{S}}}$ such that

$$\mathrm{Q}_{\mathbf{S}}(\mathbf{x}) = \sum_{i \in \widehat{\mathbf{S}}} d_i \mathbf{x}(i)^2 + \sum_{\substack{i<j \\ i,j \in \mathbf{S}}} d_{ij} \mathbf{x}(i)\mathbf{x}(j) - \sum_{i \in \mathbf{S}} d_{i0} \mathbf{x}(i)\mathbf{x}(0).$$

We fix the natural base $\left\{ \mathbf{e}_i \mid i \in \widehat{\mathbf{S}} \right\}$ in the space $\mathbb{R}^{\widehat{\mathbf{S}}}$, where $\mathbf{e}_i(j) = \delta_{ij}$ and identify a function $\mathbf{x} : \widehat{\mathbf{S}} \to \mathbb{R}$ with the vector $(x_i \mid i \in \widehat{\mathbf{S}})$, where $x_i = \mathbf{x}(i)$. For a representation $M \in \operatorname{rep} \mathbf{S}$ we define its (vector) *dimension* $\dim M \in \mathbb{R}^{\widehat{\mathbf{S}}}$ as the function $i \mapsto \dim_{\Bbbk} M(i)$. Actually, $\dim M \in \mathbb{N}^{\widehat{\mathbf{S}}}$; the latter semigroup we call the *semigroup of dimensions* for $\mathbf{S}$.

The Tits form is *integer* in the sense of [12], since $d_i \mid d_{ij}$ for all possible i, j. Therefore, (real) *roots* of this form are defined: they are vectors that can be obtained from $\mathbf{e}_i$ by a series of *reflections.* Recall that the reflection σ_i is defined as the unique non-identical linear map $\mathbb{R}^{\widehat{\mathbf{S}}} \to \mathbb{R}^{\widehat{\mathbf{S}}}$ such that $\sigma_i \mathbf{x}(j) = \mathbf{x}(j)$ for all $j \neq i$ and $\mathrm{Q}_{\mathbf{S}}(\sigma_i \mathbf{x}) = \mathrm{Q}_{\mathbf{S}}(\mathbf{x})$ for all $\mathbf{x}$. One easily sees that

$$\begin{aligned} d_0 \sigma_0 \mathbf{x}(0) &= \sum_{i \in \mathbf{S}} d_{i0} \mathbf{x}(i) - d_0 \mathbf{x}(0), \\ d_i \sigma_i \mathbf{x}(i) &= d_{i0} \mathbf{x}(0) - d_i \mathbf{x}(i) - \sum_{j \lessgtr i} d_{ij} \mathbf{x}(j) \quad \text{if } \ i \in \mathbf{S}. \end{aligned}$$

We write $\mathbf{x} > 0$ and call $\mathbf{x}$ *positive* if $\mathbf{x} \neq 0$ and $\mathbf{x}(i) \geqslant 0$ for all $i \in \widehat{\mathbf{S}}$. Especially, *positive roots* are defined. Now we are able to formulate the main theorem of our paper.

Theorem 1.5. *A WBS* $\mathbf{S}$ *is representation finite if and only if its Tits form is* weakly positive*, i.e.* $\mathrm{Q}_{\mathbf{S}}(\mathbf{x}) > 0$ *for each* $\mathbf{x} > 0$*. Moreover, in this case*

- *the dimensions of indecomposable representations of* $\mathbf{S}$ *coincide with the positive roots of the form* $\mathrm{Q}_{\mathbf{S}}$*;*
- *any two indecomposable representations having equal dimensions are isomorphic.*

The fact that representation finiteness implies weakly positivity of the Tits form is general for matrix problems. It follows, for instance, from [5]. The proof of other assertions of Theorem 1.5 relies upon *reflection functors*, which we shall construct in the next section. Though this construction was inspired by [6], its details are more complicated, so we present them thoroughly.

2. Reflection functors

First we define reflections of WBS themselves.

Definition 2.1. (1) Given a WBS $\mathbf{S}$, we set:

- $V(ii) = \mathbf{K}(i)$ and take for $\mu(iij)$ and $\mu(ijj)$ the natural isomorphisms $\mathbf{K}(i)V(ij) \simeq V(ij)$ and $V(ij)\mathbf{K}(j) \simeq V(ij)$ as soon as $V(ij)$ is defined;
- $V(ji) = V(ij)^*$ as soon as $V(ij)$ is defined;
- $\mu(kji)$ and $\mu(jik)$ to be the maps corresponding to $\mu(ikj)$ via the isomorphisms **(1)** as soon as $\mu(ikj)$ is defined.

One easily checks that the associativity conditions hold for these maps too, while the non-degeneracy conditions turn into surjectivity of the maps $\mu(j0i)$ for all $i,j \in \mathbf{S},\ j < i$.

(2) We call an element $p \in \widehat{\mathbf{S}}$ a *source* (a *sink*) if it is a maximal element of $\widehat{\mathbf{S}}^-$ (respectively, a minimal element of $\widehat{\mathbf{S}}^+$). Especially, 0 is a source (a sink) if and only if $\mathbf{S}^- = \emptyset$ (respectively, $\mathbf{S}^+ = \emptyset$).

(3) For any source or a sink p we define the *reflected* WBS $\mathbf{S}_p$ with the same underlying poset and the same values of $\mathbf{K}(i)$ as follows:

(a) If $p \in \mathbf{S}^-$ ($p \in \mathbf{S}^+$) is a source (respectively, a sink), then $\mathbf{S}_p^- = \mathbf{S}^- \setminus \{\,p\,\}$, $\mathbf{S}_p^+ = \mathbf{S}^+ \cup \{\,p\,\}$ (respectively, $\mathbf{S}_p^+ = \mathbf{S}^+ \setminus \{\,p\,\}$, $\mathbf{S}_p^- = \mathbf{S}^- \cup \{\,p\,\}$);

(b) If 0 is a source (a sink), then $\mathbf{S}^- = \mathbf{S},\ \mathbf{S}^+ = \emptyset$ (respectively, $\mathbf{S}^+ = \mathbf{S},\ \mathbf{S}^- = \emptyset$).

The new values of $V(ij)$ and $\mu(ikj)$ are defined as in item (1).

Note that if p is a source (a sink) in $\widehat{\mathbf{S}}$, it becomes a sink (respectively, a source) in $\widehat{\mathbf{S}}_p$.

We also consider the *dual* WBS.

Definition 2.2. Let $\mathbf{S}$ be a WBS, $M = (M,f)$ be a representation of $\mathbf{S}$. The *dual* WBS $\mathbf{S}^\circ$ and the *dual representation* $M^\circ(M^\circ, f^\circ)$ are defined as follows:

(1) As an ordered set, $\mathbf{S}^\circ$ is opposite to $\mathbf{S}$, i.e. consists of the same elements, but $i < j$ in $\mathbf{S}^\circ$ if and only if $j < i$ in $\mathbf{S}$. The bisection is given by the rule $\mathbf{S}^{\circ\pm} = \mathbf{S}^\mp$. The skewfields $\mathbf{K}^\circ(i)$ are opposite to $\mathbf{K}(i)$, $V^\circ(ij) = V(ji)$ as an $\mathbf{K}^\circ(i)$-$\mathbf{K}^\circ(j)$-bimodule, and $\mu^\circ(ikj) = \mu(jki)$ under the natural identification of $V^\circ(ik)V^\circ(kj)$ with $V(jk)\overgroup{V(ki)}$.

(2) $M^\circ(i) = M(i)^*$ and $f^\circ(i) = \widetilde{f(i)^*}$, namely,

(a) if $i \in \mathbf{S}^{\circ+} = \mathbf{S}^-$, then $f(i) : M(i) \to V(i0)M(0)$, thus $f(i)^* : M(0)^*V(i0)^* \to M(i)^*$ and $tif(i)^* : M(0)^* = M^\circ(0) \to M(i)^*V(i0) = V^\circ(0i)M^\circ(i)$;

(b) if $i \in \mathbf{S}^{\circ -} = \mathbf{S}^{+}$, then $f(i) : M(0) \to V(0i)M(i)$, thus $f(i)^* : M(i)^*V(0i)^* \to M(0)^*$ and $\widetilde{f(i)^*} : M(i)^* = M^\circ(i) \to M(0)^*V(0i) = V^\circ(i0)M^\circ(0)$.

(3) If $\phi \in \operatorname{Hom}_{\mathbf{S}}(M, N)$, we define $\phi^\circ : N^\circ \to M^\circ$ setting $\phi^\circ(i) = \widetilde{\phi(i)^*}$ and $\phi^\circ(ij) = \widetilde{\phi(ji)^*}$.

The following result is then evident.

Proposition 2.3. *Definition 2.2 establishes a duality functor* $^\circ : \operatorname{rep}\mathbf{S} \to \operatorname{rep}\mathbf{S}^\circ$, *i.e. an equivalence* $\operatorname{rep}\mathbf{S} \to (\operatorname{rep}\mathbf{S}^\circ)^{\mathrm{op}}$ *such that there is a natural isomorphism* $M \simeq (M^\circ)^\circ$. *Thus there is a one-to-one correspondence between indecomposable representations of* $\mathbf{S}$ *and* $\mathbf{S}^\circ$. *In particular,* $\mathbf{S}$ *is representation finite if and only if so is* $\mathbf{S}^\circ$.

We introduce some useful notations.

Definition 2.4. Let $M = (M, f)$ be a representation of a WBS $\mathbf{S}$, $p \in \mathbf{S}$. We set:

$$M^+(p) = \bigoplus_{p \leqslant i,\, i \in \mathbf{S}^+} V(pi)M(i),$$

$$M^-(p) = \bigoplus_{i \leqslant p,\, i \in \mathbf{S}^-} V(pi)M(i),$$

$$f^+(p) : V(p0)M(0) \to M^+(p) \quad \text{is the map with the components}$$

$$f^+(pi) : V(p0)M(0) \xrightarrow{1 \otimes f(i)} V(p0)V(0i)M(i) \xrightarrow{\mu(p0i)\otimes 1} V(pi)M(i),$$

$$f^-(p) : M^-(p) \to V(p0)M(0) \quad \text{is the map with the components}$$

$$f^-(pi) : V(pi)M(i) \xrightarrow{1 \otimes f(i)} V(pi)V(i0)M(0) \xrightarrow{\mu(pi0)\otimes 1} V(p0)M(0).$$

We define $M^\pm(0)$ and $f^\pm(0)$ by analogous formulae, just omitting conditions "$p \leqslant i$" and "$i \leqslant p$" under the summation sign.

Now we construct the reflection functors $\Sigma_p : \operatorname{rep}\mathbf{S} \to \operatorname{rep}\mathbf{S}_p$.

Definition 2.5. Let $M = (M, f)$ be a representations of a WBS $\mathbf{S}$, $p \in \widehat{\mathbf{S}}$ is a source or a sink. We define a representation $\Sigma_p M = (M', f')$ of the WBS $\mathbf{S}_p$ as follows (in all cases $M'(i) = M(i)$ for all $i \neq p$):

(1) If $p \in \mathbf{S}^-$ is a source, we set $f'(i) = f(i)$ for $i \neq p$, $M'(p) = \operatorname{Ker} f^+(p)/\operatorname{Im} f^-(p)$, choose a retraction $\rho_M : V(p0)M(0) \to \operatorname{Ker} f^+(p)$ and set $f'(p) = \widetilde{\pi_M \rho_M}$, where π_M is the natural surjection $\operatorname{Ker} f^+(p) \to M'(p)$.

(2) If $p = 0$ is a source, we set $M'(0) = \operatorname{Cok} f^+$ and $f'(i) = \widetilde{\pi_M(i)}$, where $\pi_M(i)$ is the i-th component of the natural surjection $\pi_M : M^+(p) \to M'(0)$.

(3) If $p \in \mathbf{S}^+$ is a sink, we set $f'(i) = f(i)$ if $i \neq p$, $M'(p) = \operatorname{Ker} f^+(p)/\operatorname{Im} f^-(p)$, choose a section $\sigma_M : \operatorname{Cok} f_p^- \to V(p0)M(0)$

and set $f'(p) = \sigma_M \varepsilon_M$, where ε_M is the natural injection $M'(p) \to \operatorname{Cok} f^-(p)$.

(4) If 0 is a sink, we set $M'(0) = \operatorname{Ker} f^-(0)$ and $f'(i) = \widetilde{\varepsilon_M(i)}$, where $\varepsilon_M(i)$ is the i-th component of the embedding $\varepsilon_M : M'(0) \to M^-(0)$.

Evidently, M' is indeed a representation of $\mathbf{S}_p$. In cases 1 and 3 these definitions depend on the choice of ρ_M and σ_M. Nevertheless, Corollary 2.7 below will show that another choice of η_M and σ_M gives isomorphic representations of $\mathbf{S}_p$.

We also define *reflected morphisms* morphisms.

Definition 2.6. Keep the notations of Definition 2.5, and let $\phi : M \to N$ be a morphism of representations, where $N = (N, g)$. We define a morphism $\Sigma_p \phi = \phi' : \Sigma_p M \to \Sigma_p(N)$ as follows (again we set $\phi'(i) = \phi(i)$ and $\phi'(ij) = \phi(ij)$ if $i \neq p$, $j \neq p$):

(1) Let $p \in \mathbf{S}^-$ be a source. Then $f^+(p)$ induces an injection $\operatorname{Im}(1 - \theta\rho_M) \to M^+(p)$, where θ is the embedding $\operatorname{Ker} f^+(p) \to V(p0)M(0)$, so we can choose a homomorphism $\xi : M^+(p) \to V(p0)M(0)$ such that $\xi f^+(p) = \theta\rho_M - 1$. We set
- $\phi'(p)(x + \operatorname{Im} f^-(p)) = (1 \otimes \phi(0))(x) + \operatorname{Im} g^-(p)$ for every $x \in \operatorname{Ker} f^+(p)$. Note that the definition of morphisms implies that $1 \otimes \phi(0)$ maps $\operatorname{Ker} f^+(p)$ to $\operatorname{Ker} g^+(p)$ and $\operatorname{Im} f^-(p)$ to $\operatorname{Im} g^-(p)$.
- $\phi'(pi) = \widetilde{\psi(i)}$, where $i > p$, $\psi(i) = \pi_N \rho_N (1 \otimes \phi(0))\xi(i)$ and $\xi(i)$ is the i-th component of ξ.

(2) Let $p = 0$ be a source. Then we choose a section $\eta : M'(0) \to M^+(0)$ and set
- $\phi'(0) = \pi_N \phi^+ \eta$, where $\phi^+ : M^+(0) \to N^+(0)$ has the (ij)-th component $1 \otimes \phi(i)$ if $i = j$, $(\mu(pji) \otimes 1)(1 \otimes \phi(ij))$ if $i < j$, and 0 if $j < i$.

(3) Let $p \in \mathbf{S}^+$ be a sink. Then $g^-(p)$ induces an surjection $N^-(p) \to \operatorname{Im}(1 - \sigma_N \tau)$, where τ is the natural surjection $V(p0)N(0) \to \operatorname{Cok} g^-(p)$, so we can choose a homomorphism $\eta : V(p0)N(0) \to N^-(p)$ such that $g^-(p)\eta = \sigma_N \tau - 1$. We set
- $\phi'(p)(x + \operatorname{Im} f^-(p)) = (1 \otimes \phi(0))(x) + \operatorname{Im} g^-(p)$ for every $x \in \operatorname{Ker} f^+(p)$.
- $\phi'(ip) = \eta(i)(1 \otimes \phi(0)) f'(p)$, where $i < p$ and $\eta(i)$ is the i-th component of η. (Recall that $f'(p) = \sigma_M \varepsilon_M$.)

(4) Let $p = 0$ be a sink. Then we choose a retraction $\xi : N^-(0) \to N'(0)$ and set
- $\phi'(0) = \xi \phi^- \varepsilon_M$, where $\phi^- : M^-(0) \to N^-(0)$ has the (ij)-th component $1 \otimes \phi(i)$ if $i = j$, $(\mu(pji) \otimes 1)(1 \otimes \phi(ij))$ if $i < j$, and 0 if $j < i$.

Again, this construction depends on the choice of ξ or η. Nevertheless, we shall show that, after some non-essential factorization, this dependence disappears.

Definition 2.7. We denote by T^p the *trivial representation* at the point p, i.e. such that $T^p(p) = \Bbbk$, $T^p(i) = 0$ for $i \neq p$, by I_p the ideal of $\operatorname{rep}\mathbf{S}$ generated by the identity morphism of T^p and by $\operatorname{rep}^{(p)}\mathbf{S}$ the factor-category $\operatorname{rep}\mathbf{S}/I_p$. We call a representation M *T^p-free* if it has no direct summands isomorphic to T^p.

The construction of $\Sigma_p M$ implies that this representation is always T^p-free. The following result is also evident.

Proposition 2.8. (1) *If $p \in \mathbf{S}^-$, M is T^p-free if and only if*

$$f(p)^{-1}\Big(\sum_{i<p} \operatorname{Im} f^-(p)(i)\Big) = 0.$$

(2) *If $p \in \mathbf{S}^+$, M is T^p-free if and only if*

$$\widetilde{f(p)}\Big(\bigcap_{i>p} \operatorname{Ker} f^+(p)(i)\Big) = M(p).$$

(3) *M is T^ω-free if and only if* $\operatorname{Ker} f^+(\omega) \subseteq \operatorname{Im} f^-(\omega)$.

Proposition 2.9. *We keep the notations of Definitions 2.5 and 2.6.*

(1) *$\Sigma_p\phi$ is indeed a morphism $\Sigma_p M \to \Sigma_p N$.*

(2) *If we choose another homomorphism ξ' or η' instead of ξ or η, satisfying the same conditions. Denote the obtained morphism $\Sigma_p M \to \Sigma_p N$ by ϕ''. Then $\phi' - \phi'' \in I_p$.*

Proof. We check the case (3); the case (1) is quite similar and the cases (2) and (4) are even easier. To prove that ϕ' is a morphism, we only have to verify that

$$g'(p)\phi'(p) = (1 \otimes \phi'(0))f'(p) + \sum_{i<p} (\mu(pi0) \otimes 1)(1 \otimes g(i))\phi'(ip).$$

First note that $\phi'(p)$ coincides with $\rho'\tau(1 \otimes \phi(0))\sigma_M\varepsilon_M$, where $\rho' : \operatorname{Cok} g^-(p) \to N'(p)$ is any retraction. Thus

$$g'(p)\phi'(p) = \sigma_N\varepsilon_N\rho'\tau(1 \otimes \phi(0))\sigma_M\varepsilon_M = \sigma_N\tau(1 \otimes \phi(0))f'(p).$$

On the other hand, $(\mu(pi0)\otimes 1)(1\otimes g(i))$ is the i-th component $g^-(p)(i)$ of $g^-(p)$. Therefore

$$(\mu(pi0) \otimes 1)(1 \otimes g(i))\phi'(ip) = g^-(p)(i)\eta(i)(1 \otimes \phi(0))f'(p) = \\ = (\sigma_N(i)\tau(i) - 1)(1 \otimes \phi(0))f'(p).$$

Thus also

$$(1 \otimes \phi'(0))f'(p) + \sum_{i<p} (\mu(pi0) \otimes 1)(1 \otimes g(i))\phi'(ip) = \sigma_N\tau(1 \otimes \phi(0))f'(p).$$

If we choose another η' such that $g^-(p)\eta' = \sigma_N\tau - 1$ then $\delta = \phi' - \phi''$ has all components zero except maybe $\delta(ip) = \gamma(i)(1 \otimes \phi(0))f'(p)$, where $\gamma = \eta - \eta'$ and $g^-(p)\gamma = 0$. Hence, $\delta = \delta'\delta''$, where $\delta'' : M' \to rT^p$ ($r = \dim_{\mathbb{K}(p)} M'(p)$) has all components zero except $\delta''(p) = 1$, while $\delta' : rT^p \to N'$ has all components zero except $\delta'(ip) = \delta(ip)$. All relations that we have to verify to show that δ' and δ'' are indeed morphisms are trivial, except the only one for δ' at the point p. But the latter coincide with the corresponding relation for δ. □

Corollary 2.10. *The constructions of subsections 2.4 and 2.5 actually defines a functor* $F_p : \operatorname{rep}^{(p)} \mathbf{S} \to \operatorname{rep}^{(p)} \mathbf{S}_p$. *In particular, the isomorphism class of* F_pM *does not depend on the choice of* ρ_M *in case 1 or* σ_M *in case 3.*

Proposition 2.11. *If* p *is a source or a sink,* $F_{pp} \simeq \mathsf{Id}$*, the identity functor of the category* $\operatorname{rep}^{(p)} \mathbf{S}$. *Therefore* $F_p : \operatorname{rep}^{(p)} \mathbf{S} \to \operatorname{rep}^{(p)} \mathbf{S}_p$ *is an equivalence.*

Proof. Again we only consider the case 1, when $p \in \mathbf{S}^-$ is a source. Let $M = (M, f)$ be a T^p-free representation of $\operatorname{rep}(\mathbf{S})$, $M' = (M', f') = F_pM$ and $M'' = (M'', f'') = F_pM'$. All components of M' and M'' coincide with those of M except $M'(p) = \operatorname{Ker} f^+(p)/\operatorname{Im} f^-(p)$, $f'(p) = \widetilde{\pi_M\rho_M}$ and $M''(p) = \operatorname{Ker} f'^+(p)/\operatorname{Im} f'^-(p)$, $f''(p) = \sigma_{M'}\varepsilon_{M'}$. By definition, $M'^+(p) = M^+(p) \oplus M'(p)$ and $f'^+(p)(p) = \pi_M\rho_M$, hence $\operatorname{Ker} f'^+(p) = \operatorname{Ker} f^+(p) \cap \operatorname{Ker} \pi_M\rho_M = \operatorname{Im} f^-(p)$. Thus $M''(p) = \operatorname{Im} f^-(p)/\sum_{i<p} \operatorname{Im} f^-(p)(i)$. By 2.3 (1), $f(p)$ is injective and $\operatorname{Im} f^-(p) = \operatorname{Im} f(p) \oplus \sum_{i<p} \operatorname{Im} f^-(p)(i)$. Therefore the natural map $\iota : M(p) \to M''(p)$ is bijective. Moreover, we can choose a section $\sigma_{M'}$ so that $\varepsilon_{M'}\sigma_{M'}$ coincides with this bijection. Then we obtain an isomorphism $\phi : M \to M''$ setting $\phi(p) = \iota$, $\phi(i) = 1$ for $i \neq p$ and $\phi(ij) = 0$ for all possible i, j. Obviously, this construction is functorial modulo the ideal I_p, so we get an isomorphism of functors $\mathsf{Id} \simeq F_{pp}$. □

Definition 2.12.

1. Let $\mathbf{p} = (p_1, p_2, \dots, p_m)$ be a sequence of elements of $\widehat{\mathbf{S}}$. We call it *admissible* and define $\mathbf{S}_\mathbf{p}$ by the following recursive rules:
   - If $m = 1$, $\mathbf{p}$ is admissible if and only if p_1 is a source or a sink; then $\mathbf{S}_\mathbf{p} = \mathbf{S}_{p_1}$.
   - If $m > 1$, $\mathbf{p}$ is admissible if and only if p_1 is a source or a sink in $\mathbf{S}_\mathbf{q}$, where $\mathbf{q} = (p_2, p_3, \dots, p_m)$; then $\mathbf{S}_p = (\mathbf{S}_\mathbf{q})_{p_1}$.
2. If p_m is a source (a sink) and, for every $k < m$, p_k is a source (respectively, a sink) in $\mathbf{S}_{(p_{k+1}, p_{k+2}, \dots, p_m)}$, we call the sequence $\mathbf{p}$ a *source sequence* (respectively, a *sink sequence*).
3. We set $\mathbf{p}^* = (p_m, p_{m-1}, \dots, p_1)$.

(4) If $\mathbf{p}$ is admissible, we denote by $\Sigma_{\mathbf{p}}$ the composition $\Sigma_{p_1}\Sigma_{p_2}\dots\Sigma_{p_m}$ and by $I_{\mathbf{p}}$ the ideal in $\operatorname{rep}\mathbf{S}$ generated by the identity morphisms of the representations $T^{(p_1,p_2,\dots,p_k)} = \Sigma_{(p_1,p_2,\dots,p_{k-1})}T^{p_k}$ $(1 \leqslant k \leqslant m)$. We set $\operatorname{rep}^{(\mathbf{p})}\mathbf{S} = \operatorname{rep}\mathbf{S}/I_{\mathbf{p}}$.

Corollary 2.13. *If a sequence* $\mathbf{p}$ *is admissible, the functor* $\Sigma_{\mathbf{p}}$ *establishes an equivalence* $\operatorname{rep}^{(\mathbf{p})}\mathbf{S} \to \operatorname{rep}^{(\mathbf{p}^*)}\mathbf{S}_{\mathbf{p}}$*, the inverse equivalence being* $\Sigma_{\mathbf{p}^*}$*. In particular, there is a one-to-one correspondence between indecomposable representations of* $\mathbf{S}$ *and* $\mathbf{S}_{\mathbf{p}}$*; thus* $\mathbf{S}$ *is representation finite if and only if so is* $\mathbf{S}_{\mathbf{p}}$*.*

3. Proof of the Main Theorem

Now we are able to prove the sufficiency in Theorem 1.5. In this section $\mathbf{S}$ denotes a WBS with a weakly positive Tits form. For any dimension vector $\mathbf{d} \in \mathbb{N}^{\widehat{\mathbf{S}}}$ we consider the set $\operatorname{rep}(\mathbf{d},\mathbf{S})$ of representations of $\mathbf{S}$ of dimension $\mathbf{d}$, namely such representations $M \in \operatorname{rep}\mathbf{S}$ that $M(i)$ is a fixed $\mathbf{K}(i)$-vector space $U(i)$ of dimension $\mathbf{d}(i)$. This set can be considered as the set of $\Bbbk$-valued points of an affine algebraic variety over $\Bbbk$. The dimension of this variety is at most

$$\mathrm{Q}^-_{\mathbf{S}}(\mathbf{d}) = \sum_{i\in\mathbf{S}} d_{i0}\mathbf{d}(i)\mathbf{d}(0) - \sum_{i\ll j} d_{ij}\mathbf{d}(i)\mathbf{d}(j).$$

Isomorphisms between these representations can be considered as $\Bbbk$-valued elements of a linear algebraic group $\mathbf{G}(\mathbf{d})$ of dimension

$$\mathrm{Q}^+_{\mathbf{S}}(\mathbf{d}) = \sum_{i\in\widehat{\mathbf{S}}} d_i\mathbf{d}(i)^2 + \sum_{i<j,\, i,j\in\mathbf{S}} d_{ij}\mathbf{d}(i)\mathbf{d}(j).$$

The isomorphism classes are just the orbits of this group. Note that $\mathrm{Q}_{\mathbf{S}} = \mathrm{Q}^+_{\mathbf{S}} - \mathrm{Q}^-_{\mathbf{S}}$. We denote by $\operatorname{ind}(\mathbf{d},\mathbf{S})$ the subset of indecomposable representations from $\operatorname{rep}(\mathbf{d},\mathbf{S})$.

In what follows we suppose that the field $\Bbbk$ is *infinite* (the case of finite fields can be then treated as in [1], and we omit the details, which are quite standard). Then one easily sees that the $\Bbbk$-valued points are dense in the variety of representations, as well as in the group $\mathbf{G}(\mathbf{d})$. Especially, if $\operatorname{rep}(\mathbf{d},\mathbf{S})$ has finitely many orbits, each component of this variety is actually a closure of some orbit. Recall that a representation M of a WBS $\mathbf{S}$ is called *sincere* if $M(i) \neq 0$ for each $i \in \widehat{\mathbf{S}}$.

We prove the sufficiency using induction on $|\mathbf{S}|$. Especially, we can suppose that $\mathbf{S}$ only has finitely many *non-sincere* indecomposable representations. More precisely, we prove the following result.

Theorem 3.1. *Let* $\mathbf{S}$ *be a WBS with weakly positive Tits form. Then*

(1) $\mathbf{S}$ *is representation finite.*
(2) $\operatorname{ind}(\mathbf{d},\mathbf{S}) \neq \emptyset$ *if and only if* $\mathbf{d}$ *is a root of the Tits form. In this case* $\operatorname{ind}(\mathbf{d},\mathbf{S})$ *consists of a unique orbit, which is dense in* $\operatorname{rep}(\mathbf{d},\mathbf{S})$*.*

(3) *If M is a sincere indecomposable representation of* $\mathbf{S}$*, there is a source (as well as a sink) sequence* $\mathbf{p}$ *such that* $M \simeq \Sigma_{\mathbf{p}} N$ *for a non-sincere representation* $N \in \operatorname{rep}(\mathbf{S}_{\mathbf{p}^*})$*.*

Our proof, like that of [6] relies on the following lemmas. (Recall that we always suppose that the Tits form is weakly positive.)

Lemma 3.2. *Suppose that the assertions of Theorem 3.1 hold for* $\mathbf{S}$*. Let p be a source or a sink in* $\widehat{\mathbf{S}}$*,* $M = (M, f) \in \operatorname{ind}(\mathbf{d}, \mathbf{S})$*, where* $\mathbf{d} \neq \mathbf{e}_p$*,* $\mathbf{d}' = \sigma_p \mathbf{d}$*. Then:*

(1) *If* $\mathbf{d}(p) > 0$*, the map* $f^+(p)$ *is surjective and the map* $f^-(p)$ *is injective.*
(2) *If* $\mathbf{d}(p) = 0$*,* $\operatorname{Ker} f^+(p) = \operatorname{Im} f^-(p)$*.*

Proof. It obviously follows from the assertion (2), since the representations satisfying the claimed conditions form an open subset in $\operatorname{rep}(\mathbf{d}, \mathbf{S})$. $\square$

Lemma 3.3. *If* $\mathbf{S}$ *is a WBS with a weakly positive Tits form, p is a source or a sink in* $\widehat{\mathbf{S}}$*,* $M \in \operatorname{ind}(\mathbf{d}, \mathbf{S})$ *and* $\mathbf{d}(p) > 0$*, then* $f^+(p)$ *is surjective and* $f^-(p)$ *is injective.*

The proof of this lemma practically coincide with that of [6, Lemma 3.3], so we omit it.

Corollary 3.4. *If* $\mathbf{S}$ *is a WBS with a weakly positive Tits form,* $M \in \operatorname{ind}(\mathbf{d}, \mathbf{S})$*,* $p \in \widehat{\mathbf{S}}$ *is a source or a sink in* $\widehat{\mathbf{S}}$ *and* $\mathbf{d}(p) > 0$*, then* $\dim \Sigma_p M = \sigma_p \dim M$*. Moreover, if N is another representation with the same properties,* $\operatorname{Hom}_{\mathbf{S}}(M, N) \simeq \operatorname{Hom}_{\mathbf{S}_p}(\Sigma_p M, \Sigma_p N)$*.*

Since the number of positive roots is finite (it follows from [4, Appendix]), Corollary 3.4 implies the assertion (3) of Theorem 3.1. Since the assertions (1) and (2) hold for non-sincere representations (by the inductive conjecture), we obtain them for all representations too. It accomplishes the proof of Theorem 3.4.

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Kyiv National Taras Shevchenko University, Department of Mechanics and Mathematics, Volodymyrska 64, 01033 Kyiv, Ukraine

E-mail address: `olena@drozd.org`