Representations of generalized bunches of chains

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1. Motivation

This paper is devoted to a class of matrix problems that is a generalization of the class considered by Nazarova and Roiter [10], Crawley-Boewey [4] and Bondarenko [2]. Even in its "original" form it was widely used in the representation theory and related topics. For example, Bondarenko used it to describe modular representations of dihedral and quasi-dihedral groups [1, 3]. Another example of application of this class of problems is the classification of Cohen-Macaulay modules over curve singularities of type P_{pq} [7] and vector bundles over projective line configurations of type $\tilde{A}_n$ [8].

Nevertheless the "original" class of problems had one fault: it did not involve extensions of the basic field that are often necessary. I am going to consider a wider class of matrix problems that I call *representations of (generalized) bunches of chains*.

2. Definitions

Prior to formulating the problem formally we need some definitions. Let's start with making clear what is the object of our research, namely what is a *bunch of chains*.

Definition 1. A bunch of chains over a field K consists of:

1. Two disjoint sets $\mathcal{E}$ and $\mathcal{F}$ (we put $\mathcal{X} := \mathcal{E} \sqcup \mathcal{F}$);

2. An ordering $<$ on $\mathcal{X}$;

3. Two symmetric relations $\sim$ and $-$ on $\mathcal{X}$ (which are *not* equivalence relations);
   We extend $-$ to an equivalence relation θ on $\mathcal{X}$, and for every class $c \in \mathcal{X}/\theta$, set $\mathcal{E}_c = c \cap \mathcal{E}$, $\mathcal{F}_c = c \cap \mathcal{F}$.

4. For any class $c \in \mathcal{X}/\theta$, two field extensions E_c, F_c of dimension at most 2 over K.
   We call an element $e \in \mathcal{E}_c$ (respectively $f \in \mathcal{F}_c$) **fat** if $F_c \neq K$ (resp. $E_c \neq K$).

These data must satisfy the following conditions:

1. If $x - y$, then $x \in \mathcal{E}$, $y \in \mathcal{F}$ or vice versa;

2. $\mathcal{E}_c$, $\mathcal{F}_c$ are chains for every $c \in \mathcal{X}/\theta$;

3. If x is fat, then $x \sim x$;

4. $\#\{y | x \sim y\} \leq 2$ for every $x \in \mathcal{X}$;

5. If $x, y, x', y' \in \mathcal{X}$ are such that $x' < x$, $y' < y$ and $x - y$, $x' - y'$, $x - y'$, then also $x' - y$.

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Let us introduce some further notations.
Definition 2.

1. We call an element $x \in \mathcal{X}$ **double** if it is not fat and $x \sim x$.

2. For every double element $x \in \mathcal{X}$ we introduce a new element x^* and set $\mathcal{X}^* = \mathcal{X} \sqcup \{x^* | x \text{ double}\}$.

3. We prolong the ordering $<$ and the relation $-$ (not $\sim$!) to $\mathcal{X}^*$ so that every new element x^* inherits all $<$ and $-$ relations that the element x has.

4. We set $\mathcal{E}_c^* = \mathcal{E}_c \sqcup \{x^* | x \in \mathcal{E}_c \text{ double}\}$, $\mathcal{F}_c^* = \mathcal{F}_c \sqcup \{x^* | x \in \mathcal{F}_c \text{ double}\}$.

Now we are ready to define what a *representation* of a bunch of chains is.
Definition 3. A representation A of $\mathcal{X}$ is given by the following data:

1. A non-negative integer n_x prescribed for every $x \in \mathcal{X}^*$ in such a way that $x \sim y$ implies $n_x = n_y$;

2. For each pair (x, y), where $x \in \mathcal{E}, y \in \mathcal{F}$ and $x - y$, a matrix A_{xy} of size $n_x \times n_y$ with entries from $E_x \otimes F_y$. (If $n_x = 0$ or $n_y = 0$, this matrix is "empty" containing no rows, respectively columns.)
   The vector $\mathbf{dim}A = \{n_x | x \in \mathcal{X}^*\}$ is called the *(vector) dimension* of the representation A.

Since we are interested in all representations of bunches of chains, we need to know which representations will be considered as "the same," namely *equivalent.* If $x \in \mathcal{E}_c^*$ ($x \in \mathcal{F}_c^*$) set $H_x = End_{E_c}(E_c \otimes F_c)$ (respectively $H_x = End_{F_c}(E_c \otimes F_c)$).

Definition 4. Two representations A and B of the same dimension are said to be **equivalent** if they can be obtained from one another by a sequence of the following *elementary transformations*:

1. Elementary transformations over E_c (or F_c) in the whole row (or column), corresponding to $x \in c$ (namely in all A_{xy} at the same time); moreover, when $x \sim x'$ and $x' \neq x$, transformations in the line corresponding to x must be the same as transformations in the line corresponding to x' (of course, if $x \in \mathcal{E}$, $x' \in \mathcal{F}$, "the same" means indeed "contragredient");

2. If $x < x'$, it is also allowed to add rows (or columns) from the line x multiplicated by homomorphisms from H_x to the rows (respectively columns) of the line x'.

For the formulation of the result we need some more definitions:
Definition 5.

1. A **word** is a sequence $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$, where $a_k \in \mathcal{X}$ and each r_k is either $\sim$ or $-$, such that for all possible values of k:
   - $a_{k-1} r_k a_k$ in $\mathcal{X}$.
   - $a_k \neq a_{k+1}$ and $r_k \neq r_{k+1}$.

2. **Admissible words** are non-symmetric words of the form:

(i) $x_1 \sim x_2 - x_3 \sim x_4 - x_5 \sim x_6 - ... - x_{n-1} \sim x_n$

(ii) $x - x_1 \sim x_2 - ... - x_{n-1} \sim x_n$
($x \nsim y$ for all $y \neq x$)

(iii) $x - x_1 \sim x_2 - ... - x_{n-1} \sim x_n - z$
($x \nsim y$ when $y \neq x$)
($z \nsim y$ when $y \neq z$)

3. A word of type (ii) is called **special** if $x \sim x$. A word of type (iii) is called **special** if $x \sim x$ but $z \nsim z$ and **bispecial** if $x \sim x$, $z \sim z$. All other words are called **ordinary**.

4. A word of type (i) such that $x_n - x_1$ is called a **cycle**. A cycle w is called **periodic** if $w = v - v - v - ... - v$ for a shorter cycle v.

Prior to giving representation of any bunch of chains, we consider the so-called *atomic problems.*

Definition 6. An **atomic problem** is a bunch of chains such that both $\mathcal{E}$ and $\mathcal{F}$ consist of one element: $\mathcal{E} = \{x\}$, $\mathcal{F} = \{y\}$, $x - y$.

3. Method of solution

Our method to calculate representations is the following:

1. To describe all representations of every atomic problem.
2. In general case, to consider an atomic part of a given bunch of chains and to reduce the corresponding matrices to normal forms.
3. To restrict elementary transformations by those that do not change the form.
4. To prove that as a result we get a new bunch of chains.

4. Atomic problems

In order to give description of the representation of any bunch of chains, let us consider some elementary cases, which we will call *"atomic problems"*.

1. $x \nsim x, y \nsim y, x \nsim y, E = F = K$.
   Then we have one matrix with entries from K:

$$\begin{array}{cc} & y \\ x & \boxed{} \end{array}$$

   and all elementary transformations are allowed.

2. $x \nsim x, y \nsim y, x \sim y, E = F = K$.
   Again, we have one matrix with entries from K:

$$\begin{array}{cc} & y \\ x & \boxed{} \end{array},$$

   but now only conjugations are allowed.

3. (a) $x \not\sim x, y \sim y, E = F = K$.
   Then we have two matrices with entries from K :

$$\begin{array}{ccc} & y & y^* \\ x & \boxed{} & \boxed{} \end{array}$$

   and elementary transformations of rows must be common in both of them.

   (b) $x \sim x, y \not\sim y, E = F = K$.
   This problem is transposed to the preceding one.

4. $x \sim x, y \sim y, E = F = K$.
   Then we have four matrices with entries from K :

$$\begin{array}{ccc} & y & y^* \\ x & \boxed{} & \boxed{} \\ x^* & \boxed{} & \boxed{} \end{array} .$$

   Elementary transformations are only allowed inside every horizontal or vertical line.

5. (a) $x \sim x$, $y \not\sim y$, $E = K$, $(F : K) = 2$ (so x is fat).
   We have a matrix with entries from F :

$$\begin{array}{cc} & F \\ K & \boxed{\mathrm{F}} \end{array} .$$

   Elementary transformations of row are allowed over K , elementary transformations of columns are allowed over F .

   (b) $x \not\sim x$, $y \sim y$, $F = K$, $(E : K) = 2$.
   This problem is transposed to the preceding one.

6. (a) $x \sim x, y \sim y, E = K, (F : K) = 2$.
   Then we have two matrices with entries from F :

$$\begin{array}{ccc} & F & F \\ K & \boxed{\mathrm{F}} & \boxed{\mathrm{F}} \end{array} .$$

   Elementary transformations of rows are common and only over K .

   (b) $x \sim x, y \sim y, (E : K) = 2, F = K$.
   This problem is transposed to the preceding one.

7. $x \sim x, y \sim y, (E : K) = (F : K) = 2$.
   Then we have one matrix wit entries from $E \otimes K$. Elementary transformations of rows (columns) are allowed over E (respectively over F).

Remark. Cases 1, 2, 3a, 3b and 4 were considered in [2].

5. Representations of atomic problems

Before giving a list of representations of the atomic problems, note that they are actually partial cases of valued graphs considered by Dlab and Ringel ([5]). So we can use their results. In particular, we use the *quadratic forms* corresponding to the atomic problems.

Non-formally speaking these forms show the difference between the number of parameters defining elementary transformations and the number of entries in matrices defining representations.

Here is the list of the quadratic forms corresponding to the atomic problems:

1. $x^2 + y^2 - xy$
2. $x^2 - x^2 = 0$
3. (a) $x^2+y^2+y_1^2-x(y_1+y_2)$ (variable y_1 corresponds to the new element y^*)
   (b) $x^2 + x_1^2 + y^2 - (x + x_1)y$
4. $x^2 + x_1^2 + y^2 + y_1^2 - (x + x_1)(y + y_1)$
5. (a) $x^2 + 2y^2 - 2xy$
   (b) $2x^2 + y^2 - 2xy$
6. (a) $x^2 + 2y^2 + 2y_1^2 - 2x(y + y_1)$
   (b) $2x^2 + 2x_1^2 + y^2 - 2(x + x_1)y$
7. $2x^2 + 2y^2 - 4xy$

Definition 9. The roots of a quadratic form $Q(\overline{x})$ corresponding to an atomic problem are defined as follows:

- *"real" roots* as solutions of the equation $Q(\overline{x}) = 1$ or, when a coefficient 2 occurs in this form, of the equation $Q(\overline{x}) = 2$;
- *"imaginary" roots* as solutions of the equation $Q(\overline{x}) = 0$.

Now we go back to the results of Dlab and Ringel ([5]). From them it follows that the dimensions of indecomposable representations are just the roots of the corresponding quadratic form and if the dimension coincides with a real root, there is only one indecomposable representation of this dimension. Moreover it is the representation of general position.

We are ready to present the list of all indecomposable representations for the atomic problems. We will skip problems number 3b, 5b, and 6b since they are simply transposed cases 3a, 5a and 6a.

To each of these representations we associate a word w and sometimes a primary polynomial $\pi(t) \in K[t], \pi(t) \neq t^d$ ("primary" always means a power of irreducible). These words will be used in the next section.

1. root $(1, 1)$:

   $\boxed{\quad 1 \quad}\,, \quad w = e - f$

2. imaginary root (n) (no real roots at all):

$$\Phi_\pi \qquad \text{(Frobenius matrix corresponding to } \pi(t)\text{)},$$

$w = e - f, \pi(t)$

and

$$J_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

$w = e \sim f - e \sim f - \ldots - e \sim f\,.$

3.
   - root $(1,1,1)$:

     $$\boxed{\;1\;} \quad \boxed{\;1\;}$$

     $w = e - f \sim f$
   - root $(1,1,0)$:

     $$\boxed{\;1\;} \quad \boxed{}$$

     (the second matrix is empty) $w = e - f$
   - root $(1,0,1)$:

     $$\boxed{} \quad \boxed{\;1\;}$$

     (the first matrix is empty) $w = e - f$

4.
   - root $(n,n,n,n+1)$

     $$\left(\begin{array}{cccc|ccccc} 1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1 \\ \hline 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 & 0 \end{array}\right)$$

     $w = f - e \sim e - f \sim f - \ldots - f \sim f$
   - root $(n,n,n,n-1)$

$$\left(\begin{array}{ccccc|cccc}
1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 1 \\
0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\
\hline
1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 1
\end{array}\right)$$

$w = e - f \sim f - e \sim e - \ldots - f \sim f$

- root $(n, n+1, n, n+1)$

$$\left(\begin{array}{cccc|ccccc}
1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\
\hline
0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\
1 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1
\end{array}\right)$$

$w = e - f \sim f - e \sim e - \ldots - f$

- imaginary root $(n, n, n, n))$

$$\left[\begin{array}{c|c} \mathrm{I} & \mathrm{I} \\ \hline \mathrm{I} & \Phi_\pi \end{array}\right]$$

$w = e - f\ ,\ \pi(t)$
and

$$\left[\begin{array}{c|c} \mathrm{I} & \mathrm{I} \\ \hline \mathrm{I} & J_0 \end{array}\right]$$

(up to permutation of horizontal and vertical lines)
$w = e \sim e - f \sim f - \ldots - f \sim f$

5.
   - root $(1,1)$:
     $(1)\ \ w = e - f$
   - root $(1,2)$:
     $(1\ \alpha)$ where $E = < 1, \alpha >$ $\ w = e - f \sim f$

6.
   - root $(2n+1, n+1, n)$

$$\left[\begin{array}{ccccc|cccc}
1 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & \dots & \vdots & \alpha & 0 & \dots & \vdots \\
\vdots & \alpha & \dots & \dots & \vdots & \vdots & 1 & \dots & \vdots \\
\vdots & \vdots & 1 & \vdots & \vdots & \vdots & \alpha & \vdots & \vdots \\
\vdots & \vdots & \alpha & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \alpha \\
0 & \dots & \dots & \dots & \alpha & 0 & \dots & \dots & 0
\end{array}\right]$$

$w = e - f \sim f - e \sim e - ... - f$

- root $(2n+1, n, n)$

$$\left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\alpha & 0 & \dots & \vdots & 1 & 0 & \dots & \vdots \\
0 & 1 & \dots & \vdots & \alpha & 0 & \dots & \vdots \\
\vdots & \alpha & \vdots & \vdots & \vdots & 1 & \dots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \alpha & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \alpha & \vdots & \vdots & \vdots & 1 \\
0 & \dots & \dots & 0 & 0 & \dots & \dots & \alpha
\end{array}\right]$$

$w = e \sim e - f \sim f - e \sim e - ... - e$

- root $(2n-1, n, n)$

$$\left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & \vdots & \alpha & 0 \dots & \vdots & \\
\vdots & \alpha & \vdots & \vdots & 0 & 1 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \alpha & \vdots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & \dots & \alpha & 0 & \dots & \dots & 1
\end{array}\right]$$

$w = e - f \sim f - e \sim e - ... - f \sim f$

- root $(2n+2, n+1, n))$

$$\left[\begin{array}{cccc|ccc}
1 & 0 & \dots & 0 & 0 & \dots & 0 \\
\alpha & 0 & \dots & \vdots & 1 & \dots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \alpha & \dots & \vdots \\
\vdots & \vdots & 1 & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \alpha & \vdots & 0 & \dots & 1 \\
0 & \dots & 0 & 1 & 0 & \dots & \alpha \\
0 & \dots & 0 & \alpha & 0 & \dots & 0
\end{array}\right]$$

$w = e \sim f - f \sim f - e \sim e - ... - f$

- root $(2n, n+1, n)$

$$\left[\begin{array}{ccccc|cccc}
1 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & 1 & \dots & \dots & \vdots & \alpha & 0 & \dots & \vdots \\
\vdots & \alpha & \dots & \dots & \vdots & 0 & 1 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \alpha & \vdots & \vdots \\
0 & \dots & \dots & 1 & 0 & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & \dots & \alpha & 0 & 0 & \dots & \dots & 1 \\
0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \alpha
\end{array}\right]$$

$w = f \sim f - e \sim e - f \sim f - ... - f$

- imaginary root $(2n, n, n)$

$$\left[\begin{array}{cccc|cccc}
\Phi_\pi & 0 & \dots & 0 & \mathrm{I} & 0 & \dots & \alpha\mathrm{I} \\
0 & \mathrm{I} & \dots & \vdots & \alpha\mathrm{I} & 0 & \dots & 0 \\
\vdots & \alpha\mathrm{I} & \vdots & \vdots & 0 & \mathrm{I} & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \alpha\mathrm{I} & \vdots & \vdots \\
0 & \dots & \dots & \mathrm{I} & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & \dots & \alpha\mathrm{I} & 0 & \dots & \dots & \mathrm{I}
\end{array}\right] \quad (n \text{ odd})$$

$$\left[\begin{array}{cccc|cccc}
\alpha\Phi_\pi & 0 & \dots & 0 & \mathrm{I} & 0 & \dots & 0 \\
0 & \mathrm{I} & \vdots & \vdots & \alpha\mathrm{I} & 0 & \dots & \vdots \\
\vdots & \alpha\mathrm{I} & \vdots & \vdots & 0 & \mathrm{I} & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \alpha\mathrm{I} & \vdots & \vdots \\
\vdots & \vdots & \vdots & \mathrm{I} & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \alpha\mathrm{I} & \vdots & \vdots & \vdots & \mathrm{I} \\
\mathrm{I} & 0 & \dots & 0 & 0 & \dots & \dots & \alpha\mathrm{I}
\end{array}\right] \quad (n \text{ even})$$

$w = e - f \sim f - e, \ \pi(t)$

7. - root $(n, n+1)$

$$\left[\begin{array}{cccccc}
\alpha & \beta & 0 & \dots & 0 & 0 \\
0 & \alpha & \beta & \dots & 0 & 0 \\
\dots & \dots & \ddots & \ddots & \dots & \dots \\
0 & 0 & 0 & \dots & \alpha & \beta
\end{array}\right]$$

where $E = < 1, \alpha >, \ F = < 1, \beta >$
$w = e - f \sim f - e \sim e - ... - f \sim f \ (n \text{ odd})$
$w = f - e \sim e - f \sim f - ... - f \sim f \ (n \text{ even})$

- root $(n+1,n)$

$$\begin{bmatrix} \alpha & 0 & \dots & 0 \\ \beta & \alpha & \dots & \vdots \\ 0 & \beta & \ddots & \vdots \\ \vdots & \vdots & \ddots & \alpha \\ 0 & \dots & \dots & \beta \end{bmatrix}$$

$w = f - e \sim e - f \sim f - ... - e \sim e$ (n odd)
$w = e - f \sim f - e \sim e - ... - e \sim e$ (n even)

- imaginary root (n,n):

$$\begin{bmatrix} \alpha I & \beta I & 0 & \dots & 0 \\ 0 & \alpha I & \beta I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta I \\ \beta\Phi_\pi & 0 & 0 & \dots & \alpha I \end{bmatrix}$$

$w = e \sim e - f \sim f$, $\pi(t)$.

6. Reduction

This section is, perhaps, the most important. It shows that when we reduce an atomic part of a bunch of chains $\mathcal{X} = \{\mathcal{E}, \mathcal{F}, <, -, \sim\}$ and restrict elementary transformations by those which do not change the canonical form of this part, we obtain again representations of a (new) bunch of chains. Namely, we can formulate the following rule to construct this new bunch of chains:

1. Choose a minimal element $e \in \mathcal{E}$ and a minimal element $f \in \mathcal{F}$ such that $e - f$. Set $\mathcal{E}^- = \mathcal{E} \setminus \{e\}$, $\mathcal{F}^- = \mathcal{F} \setminus \{f\}$.

2. Consider the atomic problem defined by the bunch $(\{e\}, \{f\})$ (with the inherited relation $\sim$); call such a problem an *atomic part* of $\mathcal{X}$.

3. Find the set $\mathcal{S}$ of all words corresponding to the indecomposable representations of this atomic problem

4. Construct the sets $\mathcal{E}^+$ and $\mathcal{F}^+$ by the following procedure:

   (a) if there is $x \in \mathcal{E}^-$ such that $x - w$ or $w - x$ is possible, but there is no $y \in \mathcal{F}^-$ with this property, then $w \in \mathcal{F}^+$; in this case set $w - x$ for all elements $x \in \mathcal{E}^-$ with this property;

   (b) if there is $x \in \mathcal{F}^-$ such that $x - w$ or $w - x$ is possible, but there is no $y \in \mathcal{E}^-$ with this property, then $w \in \mathcal{E}^+$;; in this case set $w - x$ for all elements $x \in \mathcal{F}^-$ with this property;

   (c) if there are both $x \in \mathcal{E}^-$ such that $x - w$ or $w - x$ is possible and $y \in \mathcal{F}^-$ with the same property, consider two new symbols w_e, w_f and add w_e to $\mathcal{E}^+$, w_f to $\mathcal{F}^+$; in this case set $w_e \sim w_f$, $w_e - y$ for all $y \in \mathcal{F}^-$ and $w_f - x$ for all $x \in \mathcal{E}^-$ with this property; set also $w_e \sim w_f$;

(d) proclaim w *fat* if $EndA_w/rad\,EndA_w \neq K$, where A_w is the corresponding representation;

(e) proclaim w *double* if $w \sim w$, or $w \sim w^\circ$, or $w^\circ \sim w$ where w° is the *inverse* word;

(f) for two words $w, v \in \mathcal{E}^+$ (or in $\mathcal{F}^+$), set $w < v$ if there is a non-zero homomorphism $A_w \to A_v$ (respectively $A_v \to A_w$);

(g) every word $w \in \mathcal{E}^+ \sqcup \mathcal{F}^+$ inherits all relations $<, -, \sim$ which the elements e, f had with all other elements of $\mathcal{X}$; moreover, if $w \in \mathcal{E}^+$, then $e < w$, if $w \in \mathcal{F}^+$, then $f < w$; if there are both w_e and w_f, then $e < w_e$ and $f < w_f$;

(h) delete the pair $e - f$ from the relation $-$.

5. Set $\mathcal{E}' = \mathcal{E} \sqcup \mathcal{E}^+$, $\mathcal{F}' = \mathcal{F} \sqcup \mathcal{F}^+$ and $\mathcal{X}' = \{\mathcal{E}', \mathcal{F}', <', -', \sim'\}$, where $<', -', \sim'$ denote the modified relations $<, -, \sim$.

7. Result

Theorem 1. *Let* $(\{e\}, \{f\})$ *be an atomic part of* $\mathcal{X}$, $\mathcal{X}'$ *constructed as above and* $R = \{\, A_w \,|\, w \in \mathcal{S}, w \notin \mathcal{E}^+, \mathcal{F}^+ \,\}$. *Then* $\mathcal{X}'$ *is again a bunch of chains and the indecomposable representations of* $\mathcal{X}$ *which do not belong to* R *are in one-to-one correspondence with the indecomposable representations of* $\mathcal{X}$.

Proof. The proof consists in a direct verification of this statement in each case of atomic problems described in Section 3. Therefore, we give it here only in two typical cases (the first one when the atomic problem has finitely many indecomposables, the second one when it has infinitely many indecomposables).

Case 1 (atomic problem of type 5a): $f \sim f$ is fat, $e \not\sim e$; $E = E_c = \langle\, 1, \alpha \,\rangle$, where c is the θ-class of e, $F_c = K$.

The set $\mathcal{S}$ consists of two words $v = e - f$ and $w = e - f \sim f$. The corresponding representations are:

$$A_v = (1), \qquad A_w = (1\ \alpha).$$

In this case, neither $v - x$ nor $x - v$ is possible, while $w - x$ for any element $x \in \mathcal{E}$ such that $x - f$. Hence, $\mathcal{E}^+ = \mathcal{E}$, $\mathcal{F}^+ = \mathcal{F} \sqcup \{w\}$, $f < w$ and $w < f'$ if $f < f'$.

One easily sees that if the representation A_v is in the atomic part of a representation B of $\mathcal{X}$, it splits out as a direct summand: $B \simeq A_v \oplus B'$. Hence, if B is indecomposable, then $B = A_v$. If the representation B has A_w in its atomic part, then, using elementary transformations, one can make zero the whole row, in which A_w occurs, as well as one of the two corresponding columns, say, that where α stands. Hence, the representation B is of the form:

$$B = \left|\begin{array}{ccc|c} I & \alpha I & 0 & 0 \\ 0 & 0 & 0 & B_e \\ \hline B_w & 0 & B_f & \tilde{B} \end{array}\right|,$$

where the matrix in the per left corner correspond to the pair $e-f$. One checks immediately that the following elementary transformations do not change this shape:

1. "Old" transformations of all rows (columns) not belonging to the lines under $1, \alpha$, i.e., those inherited from the original problem.
2. One can add columns of B_f multiplied by elements of E to the columns of B_w.
3. Moreover, one can add columns of B_f multiplied by $\lambda\alpha$ ($\lambda \in K$) to the columns of B_w. To do so, one has first to add this column multiplied by λ to the corresponding column under αI and then to make the latter zero with the help of the corresponding row of αI.
4. Just in the same way, if $f < f'$, one can add columns of B_f multiplied by any element from E to columns of $\tilde{B}$.

But it is just the definition of representations of the new bunch of chins $\mathcal{X}'$ and their equivalence.

Case 2 (atomic problem of type 6a): $e \sim e$ is fat, $f \sim f$ is double, $E_c = K$, $F_c = F = \langle 1, \alpha \rangle$.

Here $\mathcal{S}$ consists of the following words:

$$\begin{aligned} t_n &= e - f \sim f - e \sim e - \ldots - f, \\ u_n &= e \sim e - f \sim f - \ldots - e, \\ v_n &= e \sim e - f \sim f - \ldots - f, \\ w_n &= f \sim f - e \sim e - \ldots - e, \\ z_n &= f \sim f - e \sim e - \ldots - f. \end{aligned}$$

Denote the corresponding representation, respectively, by T_n, U_n, V_n, W_n, Z_n.

In this case $u_n, v_n \in \mathcal{E}^+$, $w_n, z_n \in \mathcal{F}^+$, while neither $t_n - x$ nor $x - t_n$ for any n. On the other hand, one can easily see that:

- if T_n occurs in the atomic part of a representation A of $\mathcal{X}$, then it splits out;
- if U_n or V_n occur, then one can make all corresponding columns and all but one (any) row zero; denote this row by $\mathbf{u}_n$ or $\mathbf{v}_n$, respectively;
- if W_n or Z_n occur, then one can make all corresponding rows and all but one (any) column zero; denote this column by $\mathbf{w}_n$ or $\mathbf{z}_n$, respectively;
- one can add the row $\mathbf{u}_n$ to the rows $\mathbf{u}_m$ and $\mathbf{v}_m$ if $m < n$, as well as the rows of $\mathbf{v}_n$ to the rows $\mathbf{u}_m$ if $m \leq n$ and to the rows $\mathbf{v}_m$ if $m < n$;
- one can add the column $\mathbf{z}_n$ to the columns $\mathbf{z}_m$ and $\mathbf{w}_m$ if $n < m$, as well as the columns of $\mathbf{w}_n$ to the columns $\mathbf{z}_m$ if $n \leq m$ and to the columns $\mathbf{w}_m$ if $n < m$;
- one can add the columns and rows corresponding to zero columns, respectively rows, of the matrix A_{ef} to all other columns, respectively rows;

- one can add the columns (respectively rows) of W_n and Z_n (respectively U_n and V_n to columns corresponding to any $f' > f$ (respectively to rows corresponding to any $e' > e$).

It means that the remaining part of the representation A can be considered just as a representation of the bunch $\mathcal{X}'$, which accomplishes the proof.

Remark. Note that this proof is completely *effective*: one can recursively deduce from it an explicit form of matrices defining all indecomposable representations of any bunch of chains.

Now we are ready to present the main theorem.

Theorem 2. *Indecomposable representations of a bunch of chains are given by the following data:*

1. **String data** *that are one of the following:*

   (i) *non-symmetric ordinary words;*

   (ii) *pairs* (w, δ) *, where* w *is a special word and* $\delta \in \{1, 2\}$ *;*

   (iii) *quadruples* $(w, m, \delta_1, \delta_2)$ *, where* w *is a non-symmetric bispecial word,* $\delta_i \in \{1, 2\}$, $m \in \mathbf{N}$ *.*

2. **Band data** *that is a pair* $(w, f(t))$ *, where* w *is a non-periodic cycle defined up to a cyclic shift and* $f(t) \neq t^d$ *is a primary polynomial over* K *(i.e. a power of an irreducible one).*

This theorem follows from the list of representations of the atomic problems and theorem 1.

7. An application

As we have mentioned, there are many possibilities of applying this class of matrix problems. We present one example that can now be obtained just as its partial case for algebras over an algebraically closed field in [6].

Corollary 3. *The category of finite length modules over a complete local noetherian ring* R *is* **tame** *if and only if* R *is isomorphic to a factor-ring of a "singularity of type"* A_1 *that is of a ring* A *that is either a dyad of two discrete valuation rings [9] or a subring of a discrete valuation ring* D *such that* $\mathrm{rad}\, A = \mathrm{rad}\, D$ *and* $D/\mathrm{rad}\, D$ *is 2-dimensional over* $A/\mathrm{rad}\, A$ *. Otherwise it is* **wild***.*

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